Optical solitons in fibre Bragg gratings with third- and fourth-order dispersive reflectivities

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Abstract. For the first time in the field of nonlinear optics, we address cubic–quartic solitons appearing in the fibre Bragg gratings with dispersive reflectivity for four different cases of nonlinear refractive-index structures. A complete spectrum of single solitons, together with some straddled solitons, emerges from the integration scheme adopted by us, which is the approach of sine–Gordon equation.

Keywords: solitons, Bragg gratings, sine-Gordon equation method

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1. Introduction

The theory of dynamics of optical solitons has left a lasting impact in telecommunications industry. There are various aspects of soliton science elaborated by the researchers. Soliton switching, intra-channel soliton collisions, quasi-monochromatic soliton dynamics, optical soliton cooling and some other topics are among the main points under interest. Another problem which is not very commonly touched upon is Bragg gratings. This is a very clever remedy to soliton transmission across intercontinental distances when a chromatic dispersion (CD) runs low. In such a crisis situation it is gratings that are introduced in an optical fibre that produces dispersive reflectivity. This ensures a stable propagation of solitons when a necessary delicate balance between dispersion and nonlinearity is maintained.

Bragg gratings have been addressed by many authors [1–26]. The existence and stability of quiescent Bragg-grating solitons have been systematically investigated in a dual-core fibre [1]. The interactions between stable quiescent Bragg-grating solitons in a dual-core system have also been reported in Ref. [2]. The interaction of quiescent gap solitons in coupled fibre Bragg gratings (FBGs) with dispersive reflectivity and cubic–quintic nonlinearity in the both cores has been addressed in Ref. [3]. The authors [4] have proposed and demonstrated experimentally a novel optimally designed FBG-filter scheme, which is based upon single fibres and designed for the conversion from RZ-OOK/DPSK/DQPSK to NRZ-OOK/DPSK/DQPSK format. A novel notch-filtering scheme, which is based on an optimally designed two-degree-of-freedom FBG, has been suggested for a transparent all-optical bit-rate NRZ-to-PRZ format conversion [5]. The existence, stability and collision dynamics of moving Bragg-grating solitons in a semi-linear dual-core system have been analyzed in Ref. [6]. The existence and stability of quiescent gap solitons in a system of two linearly coupled Bragg gratings with a cubic−quintic nonlinearity have also been studied [7]. A hybrid-core circularly cladded photonic crystal fibre has been designed and analyzed for applications in the terahertz frequency range [8]. The collisions of moving gap solitons in a system of two identical linearly coupled Bragg gratings with a cubic−quintic nonlinearity have been a subject of the study [9].

Optical solitons in the FBGs with dispersive reflectivity have been retrieved in the earlier studies [10, 11]. Bright and singular optical soliton solutions for the FBGs with dispersive reflectivity for the parabolic nonlinearity have been recovered by an extended trial-function method [12]. Bright, dark and singular solitons in these FBGs have also been revealed using a method of undetermined coefficients [13]. Optical solitons in the FBGs with dispersive reflectivity have also been obtained by the extended trial-function method for the case of quadratic–cubic nonlinearity [14]. Optical solitons in the mentioned FBGs emerge in the case of cubic–quintic–septic nonlinearity [15]. Using the extended trial function, bright and singular optical solitons have been retrieved in the same FBGs for a parabolic-nonlocal combo nonlinearity [16]. Finally, dark and singular optical solitons in the FBGs with five different forms of nonlinear refractive index have been found using a modified simple equation [17].

A system of partial differential equations for the moving optical solitons in FBGs has been studied in Ref. [18]. As follows from the sine-Gordon equation technique, a complete spectrum of single and straddled solitons emerges in the FBGs with dispersive reflectivity for five different forms of nonlinear refractive index [19]. Chirped and chirp-free solitons in the FBGs having dispersive reflectivity with a parabolic form of nonlinearity can be recovered, using a new Jacobi elliptic-function expansion [20]. Moreover, the same optical solitons have been revealed in the FBGs with dispersive reflectivity and a quadratic−cubic nonlinearity, basing on a new sub-ODE method [21]. The same sub-ODE approach has yielded in both the chirped and chirp-free optical solitons in the FBGs having dispersive reflectivity and a polynomial form of nonlinearity [22]. Furthermore, optical solitons in the same FBGs revealing the mentioned parabolic-nonlocal combo nonlinearity have been obtained via three prolific integration architectures [23]. Optical solitons in the FBGs with generalized anti-cubic nonlinearity have been retrieved from an extended auxiliary equation [24]. The authors [25] have studied the optical solitons that arise in the FBGs revealing a Kerr law for the refractive index. This is due to an extended Kudryashov's method and a new extended auxiliary-equation approach. Finally, both dark and singular optical solitons appearing in the FBGs have been addressed within the Kudryashov's model in the presence of dispersive reflectivity [26].

At present, another situation comes on board when the CD carries a low count and, hence, a balance between the CD and the nonlinearity becomes precariously low, being followed by a possible pulse collapse. In such a situation, the CD gets replaced by a combination of third-order and fourth-order dispersions. Thus, with Bragg gratings, the dual dispersion terms introduce so-called third- and fourth-order dispersive reflectivities. In the present study, we introduce the models for four different structural forms of the nonlinear refractive index, where the CD is replaced by the third-order and fourth-order dispersions together with the dispersive reflectivity. The approach of sine-Gordon equation reveals soliton solutions to these four models. A complete spectrum of solitons emerges within this integration scheme, which is discussed in this work. The details will be sketched after a brief introduction into our model.

2. Optical solitons

In this section we address a coupled nonlinear Schrodinger equation for the cases of four different forms for the nonlinear refractive index in FBGs.

2.1. Kerr law

The structure of the governing model with the Kerr nonlinear refractive index is given by

$$iq_t + ia_1 r_{xxx} + b_1 r_{xxxx} + \left(c_1 |q|^2 + d_1 |r|^2\right) q + i\alpha_1 q_x + \beta_1 r = 0, \tag{1}$$

$$ir_t + ia_2 q_{xxx} + b_2 q_{xxxx} + \left(c_2 |r|^2 + d_2 |q|^2\right) r + i\alpha_2 r_x + \beta_2 q = 0\,, \tag{2}$$

where the complex-valued functions $q(x,t)$ and $r(x, t)$ signify forward- and backward-propagating wave profiles, while x and t are the non-dimensional distance and time in dimensionless form respectively. The first terms reflect a linear temporal evolution, $i = \sqrt{-1}$, a_l and b_l $(l = 1, 2)$ imply the coefficients of respectively the third- and fourth-order dispersions, and c_l and d_l the coefficients of respectively self-phase modulation (SPM) and cross-phase modulation (XPM). Note also that a_l and β_l imply the coefficients referred respectively to inter-modal dispersion and detuning parameters.

To obtain optical solitons for the FBGs described by Eqs. (1) and (2), we assume the following travelling-wave transformations:

$$\begin{aligned} &q(x,t) = U_1(\xi) e^{i\varphi(x,t)}, \quad r(x,t) = U_2(\xi) e^{i\varphi(x,t)}, \\ &\xi = x - vt, \qquad\qquad\quad \varphi(x,t) = -\kappa x + \omega t + \theta_0, \end{aligned} \tag{3}$$

where the real-valued functions $\varphi(x,t)$ and $U_l(\xi)$ stem respectively from the phase and amplitude components of the soliton, while the real constants θ_0, κ, ω and v denote the phase constant, the wave number, the frequency and the velocity, respectively.

Substituting Eq. (3) into Eqs. (1) and (2), one obtains the real part which is a fourth-order ordinary differential equation (ODE),

$$U_{\tilde{l}}^{(iv)} + \left(3\kappa a_l - 6\kappa^2 b_l\right) U_{\tilde{l}}'' + \left(\beta_l - \kappa^3 a_l + \kappa^4 b_l\right) U_{\tilde{l}} + c_l U_l^3 + d_l U_l U_{\tilde{l}}^2 = 0\,, \tag{4}$$

and the imaginary part which is a third-order ODE,

$$\left(a_l - 4\kappa b_l\right) U_{\tilde{l}}''' + \left(\alpha_l - v\right) U_l' + \left(4\kappa^3 b_l - 3\kappa^2 a_l\right) U_{\tilde{l}}' = 0\,. \tag{5}$$

Here the following notation is used: $' = \dfrac{d}{d\xi}$, $'' = \dfrac{d^2}{d\xi^2}$, $^{(iv)} = \dfrac{d^4}{d\xi^4}$, $l = 1, 2$, and $\tilde{l} = 3 - l$. Eqs. (4) and (5) reduce to the ODE

$$b_l U_l^{(iv)} + 6\kappa^2 b_l U_l'' + \left(\kappa\alpha_l - \omega + \beta_l - 3\kappa^4 b_l\right) U_l + \left(c_l + d_l\right) U_l^3 = 0\,, \tag{6}$$

with the soliton velocity

$$v = \alpha_l - 8\kappa^3 b_l\,, \tag{7}$$

and the constraints

$$U_{\tilde{l}} = U_l\,, \tag{8}$$

$$a_l = 4\kappa b_l. \tag{9}$$

Eq. (6) holds a formal solution

$$U_l(\xi) = \sum_{i=1}^{N} \cos^{i-1}\left(V_l(\xi)\right)[B_i \sin\left(V_l(\xi)\right) + A_i \cos\left(V_l(\xi)\right] + A_0\,, \tag{10}$$

along with the ODE

$$V_l^{'}(\xi)=\sin\left(V_l(\xi)\right), \tag{11}$$

and the exact solutions

$$\begin{aligned}\sin\left(V_l(\xi)\right)&=\operatorname{sech}(\xi), \quad \sin\left(V_l(\xi)\right)=i\operatorname{csch}(\xi),\\ \cos\left(V_l(\xi)\right)&=\tanh(\xi), \quad \cos\left(V_l(\xi)\right)=\coth(\xi),\end{aligned} \tag{12}$$

where $V_l(\xi)$ is a new positive function of ξ. The balance number is given by the integer N, and A_i and B_i are constants. Balancing $U_l^{(iv)}$ with U_l^3 in Eq. (6) gives $N = 2$. Thus, Eq. (10) yields the solution

$$U_l(\xi)=A_0+B_1\sin\left(V_l(\xi)\right)+A_1\cos\left(V_l(\xi)\right)+\cos\left(V_l(\xi)\right)\left(B_2\sin\left(V_l(\xi)\right)+A_2\cos\left(V_l(\xi)\right)\right). \tag{13}$$

Inserting Eqs. (13) and (11) into Eq. (6) gives rise to the following results:

$$\begin{aligned}&\kappa=\pm\frac{\sqrt{15}}{3},\ A_0=0,\ A_1=0, A_2=0,\ B_1=0,\\ &B_2=\pm 2\sqrt{\frac{30b_l}{c_l+d_l}},\ \alpha_l=\pm\frac{\sqrt{15}\left(8b_l+3\beta_l-3\omega\right)}{15}.\end{aligned} \tag{14}$$

Substituting Eqs. (14) and (12) into Eq. (13) yields the combo dark-bright solitons

$$q(x,t)=\pm 2\sqrt{\frac{30b_1}{c_1+d_1}}\tanh\left(x-\left(\alpha_1-8\kappa^3 b_1\right)t\right)\operatorname{sech}\left(x-\left(\alpha_1-8\kappa^3 b_1\right)t\right)e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{15}$$

$$r(x,t)=\pm 2\sqrt{\frac{30b_2}{c_2+d_2}}\tanh\left(x-\left(\alpha_2-8\kappa^3 b_2\right)t\right)\operatorname{sech}\left(x-\left(\alpha_2-8\kappa^3 b_2\right)t\right)e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{16}$$

with

$$b_l\left(c_l+d_l\right)>0,$$

and the combo singular solitons

$$q(x,t)=\pm 2\sqrt{-\frac{30b_1}{c_1+d_1}}\coth\left(x-\left(\alpha_1-8\kappa^3 b_1\right)t\right)\operatorname{csch}\left(x-\left(\alpha_1-8\kappa^3 b_1\right)t\right)e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{17}$$

$$r(x,t)=\pm 2\sqrt{-\frac{30b_2}{c_2+d_2}}\coth\left(x-\left(\alpha_2-8\kappa^3 b_2\right)t\right)\operatorname{csch}\left(x-\left(\alpha_2-8\kappa^3 b_2\right)t\right)e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{18}$$

with

$$b_l\left(c_l+d_l\right)<0.$$

2.2. Parabolic law

The governing model with the parabolic nonlinear refractive index can be written as

$$iq_t+ia_1 r_{xxx}+b_1 r_{xxxx}+\left(c_1|q|^2+d_1|r|^2\right)q+\left(\lambda_1|q|^4+\mu_1|q|^2|r|^2+\zeta_1|r|^4\right)q+i\alpha_1 q_x+\beta_1 r=0, \tag{19}$$

$$ir_t+ia_2 q_{xxx}+b_2 q_{xxxx}+\left(c_2|r|^2+d_2|q|^2\right)r+\left(\lambda_2|r|^4+\mu_2|r|^2|q|^2+\zeta_2|q|^4\right)r+i\alpha_2 r_x+\beta_2 q=0, \tag{20}$$

where c_l and λ_l are the coefficients of SPM, while d_l, μ_l and ζ_l represent the coefficients of XPM. Substituting Eq. (3) into Eqs. (19) and (20) yields the real part

$$\begin{aligned}&b_l U_{\bar{l}}^{(iv)}+\left(3\kappa a_l-6\kappa^2 b_l\right)U_{\bar{l}}^{"}+\left(\kappa\alpha_l-\omega\right)U_l+\left(\beta_l-\kappa^3 a_l+\kappa^4 b_l\right)U_{\bar{l}}\\ &+c_l U_l^3+d_l U_l U_{\bar{l}}^2+\lambda_l U_l^5+\mu_l U_l^3 U_{\bar{l}}^2+\zeta_l U_l U_{\bar{l}}^4=0,\end{aligned} \tag{21}$$

and the imaginary part

$$\left(a_l-4\kappa b_l\right)U_{\tilde{l}}^{'''}+\left(\alpha_l-v\right)U_l^{'}+\left(4\kappa^3 b_l-3\kappa^2 a_l\right)U_{\tilde{l}}^{'}=0. \tag{22}$$

Eqs. (21) and (22) reduce to the ODE

$$b_l U_l^{(iv)}+6\kappa^2 b_l U_l^{''}+\left(\kappa\alpha_l-\omega+\beta_l-3\kappa^4 b_l\right)U_l+\left(c_l+d_l\right)U_l^3+\left(\lambda_l+\mu_l+\zeta_l\right)U_l^5=0, \tag{23}$$

with the constraints

$$U_{\tilde{l}}=U_l \tag{24}$$

$$a_l=4\kappa b_l, \tag{25}$$

$$v=\alpha_l-8\kappa^3 b_l. \tag{26}$$

Balancing $U_l^{(iv)}$ with U_l^5 in Eq. (23) gives $N = 1$. Thus, Eq. (10) has the solution

$$U_l(\xi)=B_1\sin\left(V_l(\xi)\right)+A_1\cos\left(V_l(\xi)\right)+A_0. \tag{27}$$

Inserting Eqs. (27) and (11) into Eq. (23) gives rise to the following results:

Case 1:

$$\begin{gathered}
A_0=0,\ B_1=0,\\
b_l=\frac{\kappa\alpha_l-\omega+\beta_l}{3\kappa^4+12\kappa^2-16},\quad A_1=\pm2\sqrt{-\frac{3\kappa^3\alpha_l-3\kappa^2\omega+3\kappa^2\beta_l-10\kappa\alpha_l+10\omega-10\beta_l}{3\kappa^4 c_l+3\kappa^4 d_l+12\kappa^2 c_l+12\kappa^2 d_l-16c_l-16d_l}},\\
\zeta_l=-\frac{1}{2\left(3\kappa^2-10\right)^2\left(\kappa\alpha_l-\omega+\beta_l\right)}\\
\times(18\kappa^5\alpha_l\lambda_l+18\kappa^5\alpha_l\mu_l-18\kappa^4\omega\lambda_l-18\kappa^4\omega\mu_l+18\kappa^4\beta_l\lambda_l+18\kappa^4\beta_l\mu_l+9\kappa^4 c_l^2\\
+18\kappa^4 c_l d_l+9\kappa^4 d_l^2-120\kappa^3\alpha_l\lambda_l-120\kappa^3\alpha_l\mu_l+120\kappa^2\omega\lambda_l+120\kappa^2\omega\mu_l-120\kappa^2\beta_l\lambda_l\\
-120\kappa^2\beta_l\mu_l+36\kappa^2 c_l^2+72\kappa^2 c_l d_l+36\kappa^2 d_l^2+200\kappa\alpha_l\lambda_l+200\kappa\alpha_l\mu_l\\
-200\omega\lambda_l-200\omega\mu_l+200\beta_l\lambda_l-48c_l^2+200\beta_l\mu_l-96c_l d_l-48d_l^2).
\end{gathered} \tag{28}$$

Substituting Eqs. (28) and (12) into Eq. (27) yields the dark solitons

$$\begin{gathered}
q(x,t)=\pm2\sqrt{-\frac{3\kappa^3\alpha_1-3\kappa^2\omega+3\kappa^2\beta_1-10\kappa\alpha_1+10\omega-10\beta_1}{3\kappa^4 c_1+3\kappa^4 d_1+12\kappa^2 c_1+12\kappa^2 d_1-16c_1-16d_1}}\\
\times\tanh\left(x-\left(\alpha_1-8\kappa^3 b_1\right)t\right)e^{i\left(-\kappa x+\omega t+\theta_0\right)},
\end{gathered} \tag{29}$$

$$\begin{gathered}
r(x,t)=\pm2\sqrt{-\frac{3\kappa^3\alpha_2-3\kappa^2\omega+3\kappa^2\beta_2-10\kappa\alpha_2+10\omega-10\beta_2}{3\kappa^4 c_2+3\kappa^4 d_2+12\kappa^2 c_2+12\kappa^2 d_2-16c_2-16d_2}}\\
\times\tanh\left(x-\left(\alpha_2-8\kappa^3 b_2\right)t\right)e^{i\left(-\kappa x+\omega t+\theta_0\right)},
\end{gathered} \tag{30}$$

and the singular solitons

$$\begin{gathered}
q(x,t)=\pm2\sqrt{-\frac{3\kappa^3\alpha_1-3\kappa^2\omega+3\kappa^2\beta_1-10\kappa\alpha_1+10\omega-10\beta_1}{3\kappa^4 c_1+3\kappa^4 d_1+12\kappa^2 c_1+12\kappa^2 d_1-16c_1-16d_1}}\\
\times\coth\left(x-\left(\alpha_1-8\kappa^3 b_1\right)t\right)e^{i\left(-\kappa x+\omega t+\theta_0\right)},
\end{gathered} \tag{31}$$

$$r(x,t) = \pm 2\sqrt{-\frac{3\kappa^3\alpha_2 - 3\kappa^2\omega + 3\kappa^2\beta_2 - 10\kappa\alpha_2 + 10\omega - 10\beta_2}{3\kappa^4 c_2 + 3\kappa^4 d_2 + 12\kappa^2 c_2 + 12\kappa^2 d_2 - 16c_2 - 16d_2}} \times\coth\left(x - \left(\alpha_2 - 8\kappa^3 b_2\right)t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{32}$$

with

$$\left(3\kappa^3\alpha_l - 3\kappa^2\omega + 3\kappa^2\beta_l - 10\kappa\alpha_l + 10\omega - 10\beta_l\right) \times\left(3\kappa^4 c_l + 3\kappa^4 d_l + 12\kappa^2 c_l + 12\kappa^2 d_l - 16c_l - 16d_l\right) < 0.$$

Case 2:

$$A_0 = 0,\ A_1 = 0,\ b_l = \frac{\kappa\alpha_l - \omega + \beta_l}{3\kappa^4 - 6\kappa^2 - 1},\quad B_1 = \pm 2\sqrt{\frac{3\kappa^3\alpha_l - 3\kappa^2\omega + 3\kappa^2\beta_l + 5\kappa\alpha_l - 5\omega + 5\beta_l}{3\kappa^4 c_l + 3\kappa^4 d_l - 6\kappa^2 c_l - 6\kappa^2 d_l - c_l - d_l}},$$

$$\begin{aligned}\zeta_l = &-\frac{1}{2\left(3\kappa^2+5\right)^2\left(\kappa\alpha_l - \omega + \beta_l\right)}(18\kappa^5\alpha_l\lambda_l + 18\kappa^5\alpha_l\mu_l - 18\kappa^4\omega\lambda_l - 18\kappa^4\omega\mu_l \\ &+18\kappa^4\beta_l\lambda_l + 18\kappa^4\beta_l\mu_l + 9\kappa^4 c_l^2 + 18\kappa^4 c_l d_l + 9\kappa^4 d_l^2 + 60\kappa^3\alpha_l\lambda_l + 60\kappa^3\alpha_l\mu_l \\ &-60\kappa^2\omega\lambda_l - 60\kappa^2\omega\mu_l + 60\kappa^2\beta_l\lambda_l + 60\kappa^2\beta_l\mu_l - 18\kappa^2 c_l^2 - 36\kappa^2 c_l d_l - 18\kappa^2 d_l^2 \\ &+50\kappa\alpha_l\lambda_l + 50\kappa\alpha_l\mu_l - 50\omega\lambda_l - 50\omega\mu_l + 50\beta_l\lambda_l + 50\beta_l\mu_l - 3c_l^2 - 6c_l d_l - 3d_l^2).\end{aligned} \tag{33}$$

Inserting Eqs. (33) and (12) into Eq. (27) yields the bright solitons

$$q(x,t) = \pm 2\sqrt{\frac{3\kappa^3\alpha_1 - 3\kappa^2\omega + 3\kappa^2\beta_1 + 5\kappa\alpha_1 - 5\omega + 5\beta_1}{3\kappa^4 c_1 + 3\kappa^4 d_1 - 6\kappa^2 c_1 - 6\kappa^2 d_1 - c_1 - d_1}} \times\text{sech}\left(x - \left(\alpha_1 - 8\kappa^3 b_1\right)t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{34}$$

$$r(x,t) = \pm 2\sqrt{\frac{3\kappa^3\alpha_2 - 3\kappa^2\omega + 3\kappa^2\beta_2 + 5\kappa\alpha_2 - 5\omega + 5\beta_2}{3\kappa^4 c_2 + 3\kappa^4 d_2 - 6\kappa^2 c_2 - 6\kappa^2 d_2 - c_2 - d_2}} \times\text{sech}\left(x - \left(\alpha_2 - 8\kappa^3 b_2\right)t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{35}$$

with

$$\left(3\kappa^3\alpha_l - 3\kappa^2\omega + 3\kappa^2\beta_l + 5\kappa\alpha_l - 5\omega + 5\beta_l\right)\times\left(3\kappa^4 c_l + 3\kappa^4 d_l - 6\kappa^2 c_l - 6\kappa^2 d_l - c_l - d_l\right) > 0,$$

and the singular solitons

$$r(x,t) = \pm 2\sqrt{\frac{3\kappa^3\alpha_1 - 3\kappa^2\omega + 3\kappa^2\beta_1 + 5\kappa\alpha_1 - 5\omega + 5\beta_1}{3\kappa^4 c_1 + 3\kappa^4 d_1 - 6\kappa^2 c_1 - 6\kappa^2 d_1 - c_1 - d_1}} \times\text{csch}\left(x - \left(\alpha_2 - 8\kappa^3 b_2\right)t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{36}$$

$$r(x,t) = \pm 2\sqrt{-\frac{3\kappa^3\alpha_2 - 3\kappa^2\omega + 3\kappa^2\beta_2 + 5\kappa\alpha_2 - 5\omega + 5\beta_2}{3\kappa^4 c_2 + 3\kappa^4 d_2 - 6\kappa^2 c_2 - 6\kappa^2 d_2 - c_2 - d_2}} \times\text{csch}\left(x - \left(\alpha_2 - 8\kappa^3 b_2\right)t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{37}$$

with

$$\left(3\kappa^3\alpha_l - 3\kappa^2\omega + 3\kappa^2\beta_l + 5\kappa\alpha_l - 5\omega + 5\beta_l\right)\times\left(3\kappa^4 c_l + 3\kappa^4 d_l - 6\kappa^2 c_l - 6\kappa^2 d_l - c_l - d\right)_l < 0.$$

Case 3:

$$A_0=0,\; b_l=\frac{\kappa\alpha_l-\omega+\beta_l}{3\kappa^4+3\kappa^2-1},\quad A_1=\pm\sqrt{-\frac{6\kappa^3\alpha_l-6\kappa^2\omega+6\kappa^2\beta_l-5\kappa\alpha_l+5\omega-5\beta_l}{6\kappa^4c_l+6\kappa^4d_l+6\kappa^2c_l+6\kappa^2d_l-2c_l-2d_l}},$$
$$B_1=\pm\sqrt{\frac{6\kappa^3\alpha_l-6\kappa^2\omega+6\kappa^2\beta_l-5\kappa\alpha_l+5\omega-5\beta_l}{6\kappa^4c_l+6\kappa^4d_l+6\kappa^2c_l+6\kappa^2d_l-2c_l-2d_l}},$$
$$\zeta_l=-\frac{1}{\left(6\kappa^2-5\right)^2\left(\kappa\alpha_l-\omega+\beta_l\right)}(36\kappa^5\alpha_l\lambda_l+36\kappa^5\alpha_l\mu_l-36\kappa^4\omega\lambda_l-36\kappa^4\omega\mu_l \tag{38}$$
$$+36\kappa^4\beta_l\lambda_l+36\kappa^4\beta_l\mu_l+18\kappa^4c_l^2+36\kappa^4c_ld_l+18\kappa^4d_l^2-60\kappa^3\alpha_l\lambda_l-60\kappa^3\alpha_l\mu_l$$
$$+60\kappa^2\omega\lambda_l+60\kappa^2\omega\mu_l-60\kappa^2\beta_l\lambda_l-60\kappa^2\beta_l\mu_l+18\kappa^2c_l^2+36\kappa^2c_ld_l+18\kappa^2d_l^2$$
$$+25\kappa\alpha_l\lambda_l+25\kappa\alpha_l\mu_l-25\omega\lambda_l-25\omega\mu_l+25\beta_l\lambda_l+25\beta_l\mu_l-6c_l^2-12c_ld_l-6d_l^2).$$

Substituting Eqs. (38) and (12) into Eq. (27) causes the combo singular solitons

$$q(x,t)=\pm\sqrt{-\frac{6\kappa^3\alpha_1-6\kappa^2\omega+6\kappa^2\beta_1-5\kappa\alpha_1+5\omega-5\beta_1}{6\kappa^4c_1+6\kappa^4d_1+6\kappa^2c_1+6\kappa^2d_1-2c_1-2d_1}}$$
$$\times\left(\coth\left(x-\left(\alpha_1-8\kappa^3b_1\right)t\right)+\operatorname{csch}\left(x-\left(\alpha_1-8\kappa^3b_1\right)t\right)\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{39}$$

$$r(x,t)=\pm\sqrt{-\frac{6\kappa^3\alpha_2-6\kappa^2\omega+6\kappa^2\beta_2-5\kappa\alpha_2+5\omega-5\beta_2}{6\kappa^4c_2+6\kappa^4d_2+6\kappa^2c_2+6\kappa^2d_2-2c_2-2d_2}}$$
$$\times\left(\coth\left(x-\left(\alpha_2-8\kappa^3b_2\right)t\right)+\operatorname{csch}\left(x-\left(\alpha_2-8\kappa^3b_2\right)t\right)\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{40}$$

with

$$\left(-6\kappa^3\alpha_l+6\kappa^2\omega-6\kappa^2\beta_l+5\kappa\alpha_l-5\omega+5\beta_l\right)$$
$$\times\left(6\kappa^4c_l+6\kappa^4d_l+6\kappa^2c_l+6\kappa^2d_l-2c_l-2d\right)_l<0.$$

Case 4:

$$\kappa=\pm\frac{\sqrt{30}}{6},\quad A_0=0,\quad A_1=\pm\sqrt{-\frac{15b_l}{c_l+d_l}},\quad B_1=\pm\sqrt{-\frac{15b_l}{c_l+d_l}},$$
$$\alpha_l=\pm\frac{\sqrt{30}\left(12\omega+133b_l-12\beta_l\right)}{60}, \tag{41}$$
$$\zeta_l=-\frac{75b_l\lambda_l+75b_l\mu_l-2c_l^2-4c_ld_l-2d_l^2}{75b_l}.$$

Inserting Eqs. (41) and (12) into Eq. (27) produces the combo dark-bright solitons

$$q(x,t)=\pm\sqrt{-\frac{15b_1}{c_1+d_1}}\left(\tanh\left(x-\left(\alpha_1-8\kappa^3b_1\right)t\right)+\operatorname{sech}\left(x-\left(\alpha_1-8\kappa^3b_1\right)t\right)\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{42}$$

$$r(x,t)=\pm\sqrt{-\frac{15b_2}{c_2+d_2}}\left(\tanh\left(x-\left(\alpha_2-8\kappa^3b_2\right)t\right)+\operatorname{sech}\left(x-\left(\alpha_2-8\kappa^3b_2\right)t\right)\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{43}$$

with

$$b_l\left(c_l+d_l\right)<0.$$

2.3. Quadratic−cubic law

The governing model with the quadratic−cubic nonlinearity of the refractive index is structured as

$$iq_t + ia_1 r_{xxx} + b_1 r_{xxxx} + c_1 q\sqrt{|q|^2 + |r|^2 + qr^* + q^* r} + \left(d_1 |q|^2 + e_1 |r|^2\right) q + \lambda_1 r^2 q^* + i\alpha_1 q_x + \beta_1 r = 0, \quad (44)$$

$$ir_t + ia_2 q_{xxx} + b_2 q_{xxxx} + c_2 r\sqrt{|r|^2 + |q|^2 + qr^* + q^* r} + \left(d_2 |r|^2 + e_2 |q|^2\right) r + \lambda_2 q^2 r^* + i\alpha_2 r_x + \beta_2 q = 0, \quad (45)$$

where d_l and e_l are the coefficients of respectively SPM and XPM, and λ_l stand for the coefficients of four-wave mixing. In the case of quadratic nonlinearity, c_l represent the coefficients of SPM and XPM, along with four-wave mixing.

Substituting Eq. (3) into Eqs. (44) and (45) leads to the real part

$$\begin{aligned} &b_l U_{\bar{l}}^{(iv)} + \left(3\kappa a_l - 6\kappa^2 b_l\right) U_{\bar{l}}'' + \left(\kappa\alpha_l - \omega\right) U_l \\ &+ \left(\kappa^4 b_l - \kappa^3 a_l + \beta_l\right) U_{\bar{l}} + c_l U_{\bar{l}}^2 + c_l U_l U_{\bar{l}} + d_l U_l^3 + \left(e_l + \lambda_l\right) U_l U_{\bar{l}}^2 = 0, \end{aligned} \quad (46)$$

and the imaginary part

$$\left(a_l - 4\kappa b_l\right) U_{\bar{l}}''' + \left(\alpha_l - v\right) U_l' + \left(4\kappa^3 b_l - 3\kappa^2 a_l\right) U_{\bar{l}}' = 0. \quad (47)$$

Eqs. (46) and (47) reduce to the ODE

$$b_l U_l^{(iv)} + 6\kappa^2 b_l U_l'' + \left(\kappa\alpha_l - \omega + \beta_l - 3\kappa^4 b_l\right) U_l + 2c_l U_l^2 + \left(d_l + e_l + \lambda_l\right) U_l^3 = 0, \quad (48)$$

with the constraints

$$U_{\bar{l}} = U_l, \quad (49)$$

$$a_l = 4\kappa b_l, \quad (50)$$

$$v = \alpha_l - 8\kappa^3 b_l. \quad (51)$$

Here Eq. (48) admits Eq. (13). Inserting Eqs. (13) and (11) into Eq. (48) gives rise to the following results:

Case 1:

$$\begin{aligned} &A_0 = \pm 2\sqrt{-\frac{30 b_l}{d_l + e_l + \lambda_l}}, \quad A_1 = 0, \quad A_2 = \pm 2\sqrt{-\frac{30 b_l}{d_l + e_l + \lambda_l}}, \\ &B_1 = 0, \quad B_2 = 0, \quad \beta_l = 3\kappa^4 b_l - 24\kappa^2 b_l - \kappa\alpha_l + \omega - 16 b_l, \\ &c_l = \pm\frac{\left(3\kappa^2 + 10\right)\sqrt{-30 b_l\left(d_l + e_l + \lambda_l\right)}}{10}. \end{aligned} \quad (52)$$

Substituting Eqs. (52) and (12) into Eq. (13) yields the dark solitons

$$q(x,t) = \pm 2\sqrt{-\frac{30 b_1}{d_1 + e_1 + \lambda_1}}\left\{1 + \tanh^2\left(x - \left(\alpha_1 - 8\kappa^3 b_1\right)t\right)\right\} e^{i\left(-\kappa x + \omega t + \theta_0\right)}, \quad (53)$$

$$r(x,t) = \pm 2\sqrt{-\frac{30 b_2}{d_2 + e_2 + \lambda_2}}\left\{1 + \tanh^2\left(x - \left(\alpha_2 - 8\kappa^3 b_2\right)t\right)\right\} e^{i\left(-\kappa x + \omega t + \theta_0\right)}, \quad (54)$$

and the singular solitons

$$q(x,t) = \pm 2\sqrt{-\frac{30 b_1}{d_1 + e_1 + \lambda_1}}\left\{1 + \coth^2\left(x - \left(\alpha_1 - 8\kappa^3 b_1\right)t\right)\right\} e^{i\left(-\kappa x + \omega t + \theta_0\right)}, \quad (55)$$

$$r(x,t)=\pm 2\sqrt{-\frac{30b_2}{d_2+e_2+\lambda_2}}\left\{1+\coth^2\left(x-\left(\alpha_2-8\kappa^3 b_2\right)t\right)\right\}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{56}$$

with

$$b_l\left(d_l+e_l+\lambda_l\right)<0\,.$$

Case 2:

$$\begin{aligned}&A_0=\pm\sqrt{\frac{11b_l}{d_l+e_l+\lambda_l}},\quad A_1=0,\quad A_2=0,\quad B_1=0,\quad B_2=\pm 2\sqrt{\frac{30b_l}{d_l+e_l+\lambda_l}},\\&\kappa=\pm\frac{\sqrt{15}}{3},\quad \alpha_l=\pm\frac{\sqrt{15}\left(3\omega+91b_l-3\beta_l\right)}{15},\quad c_l=\pm\frac{3}{2}\sqrt{11b_l\left(d_l+e_l+\lambda_l\right)}.\end{aligned} \tag{57}$$

Inserting Eqs. (57) and (12) into Eq. (13) yields the combo dark-bright solitons

$$\begin{aligned}q(x,t)=\pm&\sqrt{\frac{b_1}{d_1+e_1+\lambda_1}}\left\{\sqrt{11}+2\sqrt{30}\tanh\left(x-\left(\alpha_1-8\kappa^3 b_1\right)t\right)\operatorname{sech}\left(x-\left(\alpha_1-8\kappa^3 b_1\right)t\right)\right\}\\&\times e^{i(-\kappa x+\omega t+\theta_0)},\end{aligned} \tag{58}$$

$$\begin{aligned}r(x,t)=\pm&\sqrt{\frac{b_2}{d_2+e_2+\lambda_2}}\left\{\sqrt{11}\pm 2\sqrt{30}\tanh\left(x-\left(\alpha_2-8\kappa^3 b_2\right)t\right)\operatorname{sech}\left(x-\left(\alpha_2-8\kappa^3 b_2\right)t\right)\right\}\\&\times e^{i(-\kappa x+\omega t+\theta_0)},\end{aligned} \tag{59}$$

with the constraint

$$b_l\left(d_l+e_l+\lambda_l\right)>0.$$

Case 3:

$$\begin{aligned}&A_0=\pm\sqrt{-\frac{30b_l}{d_l+e_l+\lambda_l}},A_2=\pm\sqrt{-\frac{30b_l}{d_l+e_l+\lambda_l}},A_1=0,\\&B_1=0,B_2=\pm\sqrt{\frac{30b_l}{d_l+e_l+\lambda_l}},c_l=\pm\frac{6\kappa^2+5}{2}\sqrt{-\frac{3b_l\left(d_l+e_l+\lambda_l\right)}{10}}\,,\\&\beta_l=3\kappa^4 b_l-6\kappa^2 b_l-\kappa\alpha_l+\omega-b_l.\end{aligned} \tag{60}$$

Substituting Eqs. (60) and (12) into Eq. (13) reveals the combo singular solitons

$$\begin{aligned}q(x,t)=\pm&\sqrt{-\frac{30b_1}{d_1+e_1+\lambda_1}}\\&\times\left\{1+\coth^2\left(x-\left(\alpha_1-8\kappa^3 b_1\right)t\right)+\coth\left(x-\left(\alpha_1-8\kappa^3 b_1\right)t\right)\operatorname{csch}\left(x-\left(\alpha_1-8\kappa^3 b_1\right)t\right)\right\}\\&\times e^{i(-\kappa x+\omega t+\theta_0)},\end{aligned} \tag{61}$$

$$\begin{aligned}r(x,t)=\pm&\sqrt{-\frac{30b_2}{d_2+e_2+\lambda_2}}\\&\times\left\{1+\coth^2\left(x-\left(\alpha_2-8\kappa^3 b_2\right)t\right)+\coth\left(x-\left(\alpha_2-8\kappa^3 b_2\right)t\right)\operatorname{csch}\left(x-\left(\alpha_2-8\kappa^3 b_2\right)t\right)\right\}\\&\times e^{i(-\kappa x+\omega t+\theta_0)},\end{aligned} \tag{62}$$

with

$$b_l\left(d_l+e_l+\lambda_l\right)<0.$$

Case 4:

$$A_0 = 0,\ A_1 = 0,\ A_2 = \pm 2\sqrt{-\frac{30b_l}{d_l + e_l + \lambda_l}},\ B_2 = 0,\ B_1 = 0,$$
$$\beta_l = \frac{2\sqrt{3}\alpha_l}{3} + \omega - \frac{200b_l}{3},\ \kappa = \pm\frac{2\sqrt{3}}{3},\ c_l = \pm\sqrt{-\frac{384b_l\left(d_l + e_l + \lambda_l\right)}{5}}. \tag{63}$$

Inserting Eqs. (63) and (12) into Eq. (13) yields the dark solitons

$$q(x,t) = \pm 2\sqrt{-\frac{30b_1}{d_1 + e_1 + \lambda_1}}\tanh^2\left(x - \left(\alpha_1 - 8\kappa^3 b_1\right)t\right)e^{i(-\kappa x + \omega t + \theta_0)}, \tag{64}$$

$$r(x,t) = \pm 2\sqrt{-\frac{30b_2}{d_2 + e_2 + \lambda_2}}\tanh^2\left(x - \left(\alpha_2 - 8\kappa^3 b_2\right)t\right)e^{i(-\kappa x + \omega t + \theta_0)}, \tag{65}$$

and the singular solitons

$$q(x,t) = \pm 2\sqrt{-\frac{30b_1}{d_1 + e_1 + \lambda_1}}\coth^2\left(x - \left(\alpha_1 - 8\kappa^3 b_1\right)t\right) \times e^{i(-\kappa x + \omega t + \theta_0)}, \tag{66}$$

$$r(x,t) = \pm 2\sqrt{-\frac{30b_2}{d_2 + e_2 + \lambda_2}}\coth^2\left(x - \left(\alpha_2 - 8\kappa^3 b_2\right)t\right) \times e^{i(-\kappa x + \omega t + \theta_0)}, \tag{67}$$

with

$$b_l\left(d_l + e_l + \lambda_l\right) < 0.$$

Case 5:

$$A_0 = 0, A_1 = 0, A_2 = \pm\sqrt{-\frac{30b_l}{d_l + e_l + \lambda_l}},\ B_1 = 0,\ B_2 = \pm\sqrt{\frac{30b_l}{d_l + e_l + \lambda_l}},$$
$$\kappa = \pm\frac{2\sqrt{3}}{3}, \beta_l = \pm\frac{2\sqrt{3}\alpha_l}{3} + \omega - \frac{47b_l}{3},\ c_l = \pm\sqrt{-\frac{867b_l\left(d_l + e_l + \lambda_l\right)}{40}}. \tag{68}$$

Substituting Eqs. (68) and (12) into Eq. (13) produces the combo singular solitons

$$q(x,t) = \pm\sqrt{-\frac{30b_1}{d_1 + e_1 + \lambda_1}} \times\left(\coth^2\left(x - \left(\alpha_1 - 8\kappa^3 b_1\right)t\right) + \coth\left(x - \left(\alpha_1 - 8\kappa^3 b_1\right)t\right)\operatorname{csch}\left(x - \left(\alpha_1 - 8\kappa^3 b_1\right)t\right)\right) \times e^{i(-\kappa x + \omega t + \theta_0)}, \tag{69}$$

$$r(x,t) = \pm\sqrt{-\frac{30b_2}{d_2 + e_2 + \lambda_2}} \times\left\{\coth^2\left(x - \left(\alpha_2 - 8\kappa^3 b_2\right)t\right) + \coth\left(x - \left(\alpha_2 - 8\kappa^3 b_2\right)t\right)\operatorname{csch}\left(x - \left(\alpha_2 - 8\kappa^3 b_2\right)t\right)\right\} \times e^{i(-\kappa x + \omega t + \theta_0)}, \tag{70}$$

with $b_l\left(d_l + e_l + \lambda_l\right) < 0$.

2.4. Parabolic-nonlocal combo law

The governing system with the parabolic-nonlocal combo nonlinear refractive index is given by

$$iq_t + ia_1 r_{xxx} + b_1 r_{xxxx} + \left(c_1 |q|^2 + d_1 |r|^2\right)_{xx} q + \left(\lambda_1 |q|^4 + \mu_1 |q|^2 |r|^2 + \zeta_1 |r|^4\right) q + i\alpha_1 q_x + \beta_1 r = 0, \quad (71)$$

$$ir_t + ia_2 q_{xxx} + b_2 q_{xxxx} + \left(c_2 |r|^2 + d_2 |q|^2\right)_{xx} r + \left(\lambda_2 |r|^4 + \mu_2 |r|^2 |q|^2 + \zeta_2 |q|^4\right) r + i\alpha_2 r_x + \beta_2 q = 0, \quad (72)$$

where c_l and λ_l are the coefficients of SPM, while d_l, μ_l and ζ_l represent the coefficients of XPM.

Substituting Eq. (3) into Eqs. (71) and (72) leads to the real part

$$b_l U_{\bar{l}}^{(iv)} + \left(3\kappa a_l - 6\kappa^2 b_l\right) U_{\bar{l}}'' + \left(\kappa\alpha_l - \omega\right) U_l + \left(\beta_l - \kappa^3 a_l + \kappa^4 b_l\right) U_{\bar{l}} + 2c_l U_l \left(U_l'\right)^2 + 2d_l U_l \left(U_{\bar{l}}'\right)^2 + 2c_l U_l^2 U_l'' + 2d_l U_l U_{\bar{l}} U_{\bar{l}}'' + \lambda_l U_l^5 + \mu_l U_l^3 U_{\bar{l}}^2 + \zeta_l U_l U_{\bar{l}}^4 = 0, \quad (73)$$

and the imaginary part

$$\left(a_l - 4\kappa b_l\right) U_{\bar{l}}''' + \left(\alpha_l - v\right) U_l' + \left(4\kappa^3 b_l - 3\kappa^2 a_l\right) U_{\bar{l}}' = 0. \quad (74)$$

Eqs. (73) and (74) reduce to the ODE

$$b_l U_l^{(iv)} + 6\kappa^2 b_l U_l'' + \left(\kappa\alpha_l - \omega + \beta_l - 3\kappa^4 b_l\right) U_l + 2\left(c_l + d_l\right) U_l \left(U_l'\right)^2 + 2\left(c_l + d_l\right) U_l^2 U_l'' + \left(\lambda_l + \mu_l + \zeta_l\right) U_l^5 = 0, \quad (75)$$

with the constraints

$$U_{\bar{l}} = U_l, \quad (76)$$

$$a_l = 4\kappa b_l, \quad (77)$$

$$v = \alpha_l - 8\kappa^3 b_l. \quad (78)$$

Eq. (75) admits Eq. (27). Inserting Eqs. (27) and (11) into Eq. (75) gives rise to the following results:

Case 1:

$$A_0 = 0,\ A_1 = \pm\sqrt{\frac{3\kappa^3\alpha_l - 3\kappa^2\omega + 3\kappa^2\beta_l - 10\kappa\alpha_l + 10\omega - 10\beta_l}{6\kappa^4 c_l + 6\kappa^4 d_l + 18\kappa^2 c_l + 18\kappa^2 d_l - 12c_l - 12d_l}},$$

$$B_1 = 0,\ b_l = \frac{\kappa\alpha_l - \omega + \beta_l}{3\left(\kappa^4 + 3\kappa^2 - 2\right)},$$

$$\zeta_l = -\frac{1}{\left(3\kappa^2 - 10\right)^2 \left(\kappa\alpha_l - \omega + \beta_l\right)} \times (108\kappa^6 c_l^2 + 216\kappa^6 c_l d_l + 108\kappa^6 d_l^2 + 9\kappa^5\alpha_l\lambda_l + 9\kappa^5\alpha_l\mu_l - 9\kappa^4\omega\lambda_l - 9\kappa^4\omega\mu_l + 9\kappa^4\beta_l\lambda_l + 9\kappa^4\beta_l\mu_l + 252\kappa^4 c_l^2 + 504\kappa^4 c_l d_l + 252\kappa^4 d_l^2 - 60\kappa^3\alpha_l\lambda_l - 60\kappa^3\alpha_l\mu_l + 60\kappa^2\omega\lambda_l + 60\kappa^2\omega\mu_l - 60\kappa^2\beta_l\lambda_l - 60\kappa^2\beta_l\mu_l - 432\kappa^2 c_l^2 - 864\kappa^2 c_l d_l - 432\kappa^2 d_l^2 + 100\kappa\alpha_l\lambda_l + 100\kappa\alpha_l\mu_l - 100\omega\lambda_l - 100\omega\mu_l + 100\beta_l\lambda_l + 100\beta_l\mu_l + 144c_l^2 + 288c_l d_l + 144d_l^2). \quad (79)$$

Substituting Eqs. (79) and (12) into Eq. (27) causes the dark solitons

$$q(x,t)=\pm\sqrt{\frac{3\kappa^3\alpha_1-3\kappa^2\omega+3\kappa^2\beta_1-10\kappa\alpha_1+10\omega-10\beta_1}{6\kappa^4c_1+6\kappa^4d_1+18\kappa^2c_1+18\kappa^2d_1-12c_1-12d_1}}\times\tanh\left(x-\left(\alpha_1-8\kappa^3b_1\right)t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{80}$$

$$r(x,t)=\pm\sqrt{\frac{3\kappa^3\alpha_2-3\kappa^2\omega+3\kappa^2\beta_2-10\kappa\alpha_2+10\omega-10\beta_2}{6\kappa^4c_2+6\kappa^4d_2+18\kappa^2c_2+18\kappa^2d_2-12c_2-12d_2}}\times\tanh\left(x-\left(\alpha_2-8\kappa^3b_2\right)t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{81}$$

and the singular solitons

$$q(x,t)=\pm\sqrt{\frac{3\kappa^3\alpha_1-3\kappa^2\omega+3\kappa^2\beta_1-10\kappa\alpha_1+10\omega-10\beta_1}{6\kappa^4c_1+6\kappa^4d_1+18\kappa^2c_1+18\kappa^2d_1-12c_1-12d_1}}\times\coth\left(x-\left(\alpha_1-8\kappa^3b_1\right)t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{82}$$

$$r(x,t)=\pm\sqrt{\frac{3\kappa^3\alpha_2-3\kappa^2\omega+3\kappa^2\beta_2-10\kappa\alpha_2+10\omega-10\beta_2}{6\kappa^4c_2+6\kappa^4d_2+18\kappa^2c_2+18\kappa^2d_2-12c_2-12d_2}}\times\coth\left(x-\left(\alpha_2-8\kappa^3b_2\right)t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{83}$$

with

$$\left(3\kappa^3\alpha_l-3\kappa^2\omega+3\kappa^2\beta_l-10\kappa\alpha_l+10\omega-10\beta_l\right)\times\left(6\kappa^4c_l+6\kappa^4d_l+18\kappa^2c_l+18\kappa^2d_l-12c_l-12d_l\right)>0.$$

Case 2:

$$\begin{aligned}
&A_0=0,\ A_1=0,\\
&b_l=\frac{\kappa\alpha_l-\omega+\beta_l}{3\kappa^4-6\kappa^2-1},\\
&B_1=\pm\sqrt{\frac{3\kappa^3\alpha_l-3\kappa^2\omega+3\kappa^2\beta_l+5\kappa\alpha_l-5\omega+5\beta_l}{3\kappa^4c_l+3\kappa^4d_l-6\kappa^2c_l-6\kappa^2d_l-c_l-d_l}},\\
&\zeta_l=\frac{1}{\left(3\kappa^2+5\right)^2\left(\kappa\alpha_l-\omega+\beta_l\right)}\times\\
&(54\kappa^6c_l^2+108\kappa^6c_ld_l+54\kappa^6d_l^2-9\kappa^5\alpha_l\lambda_l-9\kappa^5\alpha_l\mu_l+9\kappa^4\omega\lambda_l+9\kappa^4\omega\mu_l-9\kappa^4\beta_l\lambda_l\\
&-9\kappa^4\beta_l\mu_l-90\kappa^4c_l^2-180\kappa^4c_ld_l-90\kappa^4d_l^2-30\kappa^3\alpha_l\lambda_l-30\kappa^3\alpha_l\mu_l+30\kappa^2\omega\lambda_l\\
&+30\kappa^2\omega\mu_l-30\kappa^2\beta_l\lambda_l-30\kappa^2\beta_l\mu_l-54\kappa^2c_l^2-108\kappa^2c_ld_l-54\kappa^2d_l^2-25\kappa\alpha_l\lambda_l\\
&-25\kappa\alpha_l\mu_l+25\omega\lambda_l+25\omega\mu_l-25\beta_l\lambda_l-25\beta_l\mu_l-6c_l^2-12c_ld_l-6d_l^2).
\end{aligned} \tag{84}$$

Inserting Eqs. (84) and (12) into Eq. (27) yields the bright solitons

$$q(x,t)=\pm\sqrt{\frac{3\kappa^3\alpha_1-3\kappa^2\omega+3\kappa^2\beta_1+5\kappa\alpha_1-5\omega+5\beta_1}{3\kappa^4c_1+3\kappa^4d_1-6\kappa^2c_1-6\kappa^2d_1-c_1-d_1}}\times\operatorname{sech}\left(x-\left(\alpha_1-8\kappa^3b_1\right)t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{85}$$

$$r(x,t)=\pm\sqrt{\frac{3\kappa^3\alpha_2-3\kappa^2\omega+3\kappa^2\beta_2+5\kappa\alpha_2-5\omega+5\beta_2}{3\kappa^4c_2+3\kappa^4d_2-6\kappa^2c_2-6\kappa^2d_2-c_2-d_2}}\times\operatorname{sech}\left(x-\left(\alpha_2-8\kappa^3b_2\right)t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{86}$$

with

$$\left(3\kappa^3\alpha_l-3\kappa^2\omega+3\kappa^2\beta_l+5\kappa\alpha_l-5\omega+5\beta_l\right)\times\left(3\kappa^4c_l+3\kappa^4d_l-6\kappa^2c_l-6\kappa^2d_l-c_l-d_l\right)>0,$$

and the singular solitons

$$q(x,t)=\pm\sqrt{-\frac{3\kappa^3\alpha_1-3\kappa^2\omega+3\kappa^2\beta_1+5\kappa\alpha_1-5\omega+5\beta_1}{3\kappa^4c_1+3\kappa^4d_1-6\kappa^2c_1-6\kappa^2d_1-c_1-d_1}}\times\operatorname{csch}\left(x-\left(\alpha_1-8\kappa^3b_1\right)t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{87}$$

$$r(x,t)=\pm\sqrt{-\frac{3\kappa^3\alpha_2-3\kappa^2\omega+3\kappa^2\beta_2+5\kappa\alpha_2-5\omega+5\beta_2}{3\kappa^4c_2+3\kappa^4d_2-6\kappa^2c_2-6\kappa^2d_2-c_2-d_2}}\times\operatorname{csch}\left(x-\left(\alpha_2-8\kappa^3b_2\right)t\right)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{88}$$

with

$$\left(3\kappa^3\alpha_l-3\kappa^2\omega+3\kappa^2\beta_l+5\kappa\alpha_l-5\omega+5\beta_l\right)\times\left(3\kappa^4c_l+3\kappa^4d_l-6\kappa^2c_l-6\kappa^2d_l-c_l-d_l\right)<0.$$

Case 3:

$$\begin{aligned}
&A_0=0,\\
&b_l=\frac{8\left(\kappa\alpha_l-\omega+\beta_l\right)}{3\left(8\kappa^4+6\kappa^2-1\right)},\\
&A_1=\pm\sqrt{\frac{12\kappa^3\alpha_l-12\kappa^2\omega+12\kappa^2\beta_l-10\kappa\alpha_l+10\omega-10\beta_l}{24\kappa^4c_l+24\kappa^4d_l+18\kappa^2c_l+18\kappa^2d_l-3c_l-3d_l}},\\
&B_1=\pm\sqrt{-\sqrt{\frac{12\kappa^3\alpha_l-12\kappa^2\omega+12\kappa^2\beta_l-10\kappa\alpha_l+10\omega-10\beta_l}{24\kappa^4c_l+24\kappa^4d_l+18\kappa^2c_l+18\kappa^2d_l-3c_l-3d_l}}},\\
&\zeta_l=-\frac{1}{4\left(6\kappa^2-5\right)^2\left(\kappa\alpha_l-\omega+\beta_l\right)}\\
&\times(432\kappa^6c_l^2+864\kappa^6c_ld_l+432\kappa^6d_l^2+144\kappa^5\alpha_l\lambda_l\\
&+144\kappa^5\alpha_l\mu_l-144\kappa^4\omega\lambda_l-144\kappa^4\omega\mu_l+144\kappa^4\beta_l\lambda_l+144\kappa^4\beta_l\mu_l+252\kappa^4c_l^2\\
&+504\kappa^4c_ld_l+252\kappa^4d_l^2-240\kappa^3\alpha_l\lambda_l-240\kappa^3\alpha_l\mu_l+240\kappa^2\omega\lambda_l+240\kappa^2\omega\mu_l\\
&-240\kappa^2\beta_l\lambda_l-240\kappa^2\beta_l\mu_l-108\kappa^2c_l^2-216\kappa^2c_ld_l-108\kappa^2d_l^2+100\kappa\alpha_l\lambda_l+100\kappa\alpha_l\mu_l\\
&-100\omega\lambda_l-100\omega\mu_l+100\beta_l\lambda_l+100\beta_l\mu_l+9c_l^2+18c_ld_l+9d_l^2).
\end{aligned} \tag{89}$$

Substituting Eqs. (89) and (12) into Eq. (27) yields the combo singular solitons

$$q(x,t)=\pm\sqrt{\frac{12\kappa^3\alpha_1-12\kappa^2\omega+12\kappa^2\beta_1-10\kappa\alpha_1+10\omega-10\beta_1}{24\kappa^4c_1+24\kappa^4d_1+18\kappa^2c_1+18\kappa^2d_1-3c_1-3d_1}}$$
$$\times\left\{\coth\left(x-\left(\alpha_1-8\kappa^3b_1\right)t\right)+\operatorname{csch}\left(x-\left(\alpha_1-8\kappa^3b_1\right)t\right)\right\} \tag{90}$$
$$\times e^{i\left(-\kappa x+\omega t+\theta_0\right)},$$

$$r(x,t)=\pm\sqrt{\frac{12\kappa^3\alpha_2-12\kappa^2\omega+12\kappa^2\beta_2-10\kappa\alpha_2+10\omega-10\beta_2}{24\kappa^4c_2+24\kappa^4d_2+18\kappa^2c_2+18\kappa^2d_2-3c_2-3d_2}}$$
$$\times\left\{\coth\left(x-\left(\alpha_2-8\kappa^3b_2\right)t\right)+\operatorname{csch}\left(x-\left(\alpha_2-8\kappa^3b_2\right)t\right)\right\} \tag{91}$$
$$\times e^{i\left(-\kappa x+\omega t+\theta_0\right)},$$

with

$$\left(12\kappa^3\alpha_l-12\kappa^2\omega+12\kappa^2\beta_l-10\kappa\alpha_l+10\omega-10\beta_l\right)$$
$$\times\left(24\kappa^4c_l+24\kappa^4d_l+18\kappa^2c_l+18\kappa^2d_l-3c_l-3d_l\right)>0.$$

3. Conclusion

In the present study, we have recovered the solutions in the forms of bright, dark and singular optical solitons for the cases of third- and fourth-order dispersive reflectivities, as opposed to a common norm of dispersive reflectivity with the CD. This very concept and the corresponding results have been reported and analyzed for the first time in the field of nonlinear optics. Hence, these results are truly novel. The results obtained in this work and the new technicalities developed by us would enhance both a general soliton science and a dynamics of solitons propagating through optical fibres across intercontinental distances. The foundation stones for such a new concept are thus grounded.

What needs to be done next is expanding and extending the ideas mentioned above to various additional scenarios. The latter would include retrieval of the conservation laws within the model, addressing the model with fractional temporal evolutions and studying the appropriate model with stochasticity (see, e.g., the stochastic coefficients introduced in Refs. [27−30]). In addition to the analytical approaches, the models developed in the present work can be handled numerically. Moreover, the known Adomian and Laplace–Adomian decomposition schemes, the finite-element approach and some other techniques can be implemented. Such research activities are now in progress.

Disclosure

The authors declare no conflict of interest.

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Анотація. *Вперше в нелінійній оптиці розглянуто кубічно-квартичні солітони, які з'являються у волоконних бреггівських ґратках з дисперсійною відбивною здатністю для чотирьох різних випадків структур з нелінійним показником заломлення. Із прийнятої нами схеми інтегрування, яка є підходом рівняння синус–Гордона, випливає повний спектр одиничних солітонів, разом із деякими розмежованими солітонами.*