Laguerre-Gaussian Beams in Uniaxial Crystals

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Abstract

We described propagation of light beams in uniaxial crystals. We solved the paraxial wave equation and find a full set of Laguerre-Gaussian beams with a complex argument. We showed that any light beam could be expressed by the superposition of the mode beams in the crystal. We found the conformity between the presented theory and the representations of the fields in other theories and showed that conversion of the Laguerre-Gaussian beams with the indices $n=0$ (and m being arbitrary) can be considered on the base of Jones's matrix formalism.

Key words: optical vortex, uniaxial crystal, singular beam

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Most lasers oscillate with a transverse electric field distribution that can be rather accurately described in terms of the paraxial Laguerre-Gaussian beams [1]. These mode beams are structurally stable solutions to Maxwell's equations. It means that a beam field on propagation remains without any changing if we ignore its radial scaling. At the same time, on propagation of the similar beams in uniaxial crystals their topological field structure can experience essential transformations as a result of the events of the birth and death of the phase singularities [2, 3]. To present day, the basic method of the field description in crystals is founded on the plane wave representation: the given propagation direction is put in accordance to a plane wave. The field is represented by the angular-spectral integral over all propagation directions (see, e.g., [3]). In the most practically important cases, however, the spectral integral cannot be found in the exact form while important traits of a singular beam can be lost in the result of the approximate integration. This deficiency vanishes in the method of the Laguerre-Gaussian beams.

Recently we have shown [2] that the circularly polarized Gaussian beam propagating along the optical axis in a uniaxial crystal gained absolutely new properties not inherent in the initial one. In fact, there appears the orthogonally polarized vortex-bearing beam in the light flow emitted from the crystal. The optical vortex [4] in the induced singular beam has a double topological charge. Its sign is defined by the direction of the electric vector circulation in the initial beam. Although the method enables us to use the simple Jones's matrix formalism to calculate complex optical devices, it is an approximate one. Besides, the authors didn't define the limits of applicability of this technique.

In the given article we present the exact solution of the paraxial wave equation in terms of the Laguerre-Gaussian beams with the complex argument in uniaxial crystals and define the link between the given method and the approximate method in the article [2].

Consider an anisotropic medium with the

permittivity tensor in the form: $\hat{\varepsilon} = diag\{\varepsilon_{ii}\}$, $j = 1,2,3$ и $\varepsilon_{11} = \varepsilon_{22} = \varepsilon$, $\varepsilon_{33} = \varepsilon_3$. If a paraxial light beam propagates along the z-axis the vector wave equation can be rewritten as

$$\left(\nabla^2 + k_0^2 \hat{\boldsymbol{\varepsilon}}\right)\mathbf{E} = \alpha \nabla\left(\nabla_t \mathbf{E}_t\right), \qquad (1)$$

where k_0 is the wave vector in free space, $\alpha = \Delta\varepsilon / \varepsilon$, $\Delta\varepsilon = \varepsilon - \varepsilon_3$. As far as we deal with the paraxial beams so that $k z_0 >> 1$ ($z_0 = k\rho^2/2$, $k = k_0\sqrt{\varepsilon}$ and ρ is the beam waist radius at $z = 0$) then eq. (1) can be transformed into the paraxial wave equation

$$\left(\nabla_t^2 + 2ik\partial_z\right)\widetilde{\mathbf{E}}_t = -\frac{\Delta\varepsilon}{\varepsilon}\nabla_t\left(\nabla_t \widetilde{\mathbf{E}}_t\right), \qquad (2)$$

where $\mathbf{E}_t = \widetilde{\mathbf{E}}_t(x,y,z)\exp\left(i\widetilde{k}z\right)$ and $\widetilde{\mathbf{E}}_t$ stands for the transverse electric vector $\widetilde{\mathbf{E}}_t = \{E_x \quad E_y\}$ while the longitudinal E_z component can be found from the equation

$$E_z \approx -\frac{1}{ik}\nabla_t \widetilde{\mathbf{E}}_t. \qquad (3)$$

being the consequence of the equation $\nabla_t \mathbf{E} = 0$ and the paraxial condition $k z_0 >> 1$.

The solutions of eq. (2) are

$$\widetilde{\mathbf{E}}_t = \nabla_t \Psi, \qquad (4)$$

$$\mathbf{E}_t = \nabla \times (\mathbf{a}\Psi), \qquad (5)$$

where the wavefunction Ψ satisfies the paraxial wave equation

$$\left(\nabla_t^2 + 2i\beta\frac{\partial}{\partial z}\right)\Psi = 0, \qquad (6)$$

where β is some wavenumber and $\mathbf{a}$ is the vector with the components $\{a_x \quad a_y \quad a_z\}$.

Choose the vector $\mathbf{a}$ in the form $\mathbf{a} = \{0,0,1\}$ and

$$\Psi = \frac{1}{\sigma}\exp\left(\frac{r^2}{\rho^2\sigma}\right). \qquad (7)$$

Then the solution of the paraxial wave equation (2) can be found from eq. (5) as

$$\begin{aligned} |1 \quad 1\rangle_{TE} &= -\frac{2}{\rho^2\sigma_E}\left(\hat{\mathbf{x}}y - \hat{\mathbf{y}}x\right)\Psi_E = \\ &= i\frac{r}{\rho^2\sigma_E}\left[\hat{\mathbf{c}}^+\exp(-i\varphi) - \hat{\mathbf{c}}^-\exp(i\varphi)\right]\Psi_E \end{aligned}, \qquad (8)$$

where $\hat{\mathbf{c}}^+ = \hat{\mathbf{x}} + i\hat{\mathbf{y}}$, $\hat{\mathbf{c}}^- = \hat{\mathbf{x}} - i\hat{\mathbf{y}}$ are the unit vectors in the circularly polarized basis, $\sigma_E = 1 + iz/z_{0E} = \sigma$ so that $z_{0e} = z_0$ and the wavenumber of $|1 \quad 1\rangle_{TE}$ mode beam is

$$\beta = k_E = k. \qquad (9)$$

Besides, we find the second solution of eq.(2) from eqs. (4) and (7):

$$\begin{aligned} |1 \quad 1\rangle_{TM} &= \frac{2}{\rho^2\sigma_M}\left(\hat{\mathbf{x}}x + \hat{\mathbf{y}}y\right)\Psi_M = \\ &= \frac{r}{\rho^2\sigma_M}\left[\hat{\mathbf{c}}^+\exp(-i\varphi) + \hat{\mathbf{c}}^-\exp(i\varphi)\right]\Psi_M \end{aligned}, \qquad (10)$$

where $\sigma_M = 1 + iz/z_{0M}$, $z_{0M} = k_M\rho^2/2$, $\Psi_M = \exp\left(-\frac{r^2}{\rho^2\sigma_M}\right)/\sigma_M$ and the wavenumber of $|1 \quad 1\rangle_{TM}$ mode beam is

$$\beta = k_M = \frac{k_E}{1-\alpha} = k\frac{\varepsilon_3}{\sqrt{\varepsilon}}. \qquad (11)$$

In order to find the full set of the mode beams we form the operator

$$\hat{L}_{n,l}^{(\pm)} = \frac{\partial^n}{\partial z^n}\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right)^l. \qquad (12)$$

As far as the operator $\hat{L}_{n,l}^{(\pm)}$ commutates with the operator $\hat{D} = \nabla_t^2 + 2ik\frac{\partial}{\partial z}$, then the functions

$$|n \quad m\rangle^{(\pm)} = \hat{L}_{n,l}^{(\pm)}\Psi \qquad (13)$$

are also the solutions of the eq. (2) without the right part. The Laguerre-Gaussian beam can be obtained from eq. (13) as

$$\begin{aligned} |n \quad m\rangle &= (-1)^{n+m}\frac{2^{n+m}i^n n!}{(\rho\sigma)^{n+1}}\left(\frac{r}{\rho\sigma}\right)^m \times \\ &\times L_n^m\left(\frac{r^2}{\rho^2\sigma}\right)\exp\left(-\frac{r^2}{\rho^2\sigma}\right)\exp(\pm im\varphi) \end{aligned}. \qquad (14)$$

where $L_n^m(x)$ is a Laguerre's polynomial.

The right part of the eq. (2) transforms essentially the solutions (14). Now we have to find the solution in the form

$$\left|n' \quad m'\right\rangle^{(\pm)}_{TE,TM} = \hat{L}^{(\pm)}_{n,l}\begin{Bmatrix}\left|1 \quad 1\right\rangle_{TE}\\ \left|1 \quad 1\right\rangle_{TM}\end{Bmatrix}. \tag{15}$$

Using eqs (8), (10) and (15), after some mathematics we find

$$\left|n+1 \quad l+1\right\rangle^{(+)}_{TE} =$$

$$=A_E \frac{r^{l-1}\exp(i(l-1)\varphi)}{\rho^{2l}\sigma_E^{n+l+1}}\times\left\{\hat{\mathbf{c}}^+(n+1)L^{(l-1)}_{n+1}\left(\frac{r^2}{\rho^2\sigma_E}\right)+\hat{\mathbf{c}}^-\frac{r^2}{\rho^2\sigma_E}L^{(l+1)}_n\left(R_E^2\right)\exp(i2\varphi)\right\}\times\exp\left(-\frac{r^2}{\rho^2\sigma_E}\right), \tag{16}$$

$$\left|n+1 \quad l+1\right\rangle^{(-)}_{TE} =$$

$$= A_E \frac{r^{l-1}\exp(-i(l-1)\varphi)}{\rho^{2l}\sigma_E^{n+l+1}}\left\{\hat{\mathbf{c}}^+\frac{r^2}{\rho^2\sigma_E}L^{(l+1)}_n\left(\frac{r^2}{\rho^2\sigma_E}\right)\exp(-i2\varphi)+\hat{\mathbf{c}}^-(n+1)L^{(l-1)}_{n+1}\left(\frac{r^2}{\rho^2\sigma_E}\right)\right\}\exp\left(-\frac{r^2}{\rho^2\sigma_E}\right), \tag{17}$$

$$\left|n+1 \quad l+1\right\rangle^{(+)}_{TM} =$$

$$= A_M \frac{r^{l-1}\exp(i(l-1)\varphi)}{\rho^{2l}\sigma_M^{n+l+1}}\times\left\{\hat{\mathbf{c}}^+(n+1)L^{(l-1)}_{n+1}\left(\frac{r^2}{\rho^2\sigma_M}\right)-\hat{\mathbf{c}}^-\frac{r^2}{\rho^2\sigma_M}L^{(l+1)}_n\left(\frac{r^2}{\rho^2\sigma_M}\right)\exp(i2\varphi)\right\}\times\exp\left(-\frac{r^2}{\rho^2\sigma_M}\right), \tag{18}$$

$$\left|n+1 \quad l+1\right\rangle^{(-)}_{TM} =$$

$$= A_M \frac{r^{l-1}\exp(-i(l-1)\varphi)}{\rho^{2l}\sigma_M^{n+l+1}}\times\left\{\hat{\mathbf{c}}^+\frac{r^2}{\rho^2\sigma_M}L^{(l+1)}_n\left(\frac{r^2}{\rho^2\sigma_M}\right)e^{-i2\varphi}-\hat{\mathbf{c}}^-(n+1)L^{(l-1)}_{n+1}\left(\frac{r^2}{\rho^2\sigma_M}\right)\right\}\times\exp\left(-\frac{r^2}{\rho^2\sigma_M}\right), \tag{19}$$

where

$$A_E = \frac{(-2)^{n+l}i^{n+1}n!}{z_{0E}^n},\quad A_M = \frac{(-2)^{n+l}i^n n!}{z_{0M}^n}$$

and $n,m>0$. The beams with indices $n=0$ and $m=0$ are set by eqs.(8) and (10).

However, the expressions obtained do not describe the light wave transformations for the $\left|n' \quad 0\right\rangle$ - Laguerre- Gaussian beams. In order to find two missing beam groups $\left|n' \quad 0\right\rangle_{TE}$ and $\left|n' \quad 0\right\rangle_{TM}$ it remarks that the expressions

$$\left|0 \quad 1\right\rangle_{TE} = \int\left|1 \quad 1\right\rangle_{TE}dz = i\frac{z_{0E}}{r}\times\left(\hat{\mathbf{c}}^+\exp(-i\varphi)-\hat{\mathbf{c}}^-\exp(i\varphi)\right)\times\exp\left(-\frac{r^2}{\rho^2\sigma_E}\right), \tag{20}$$

$$\left|0 \quad 1\right\rangle_{TM} = \int\left|1 \quad 1\right\rangle_{TM}dz = \frac{z_{0M}}{r}\times\left(\hat{\mathbf{c}}^+\exp(-i\varphi)+\hat{\mathbf{c}}^-\exp(i\varphi)\right)\times\exp\left(-\frac{r^2}{\rho^2\sigma_M}\right) \tag{21}$$

are also the solutions of eq. (2). From eqs (12), (20) and (21) provided that $n'=0$ we find the missing beam groups

$$\left|0 \quad l+1\right\rangle^{+}_{TE} == B_E\left\{\hat{\mathbf{c}}^+\frac{r^{l-1}\exp(i(l-1)\varphi)}{\left(\rho^2\sigma_E\right)^l}+\hat{\mathbf{c}}^-\sum_{j=0}^{l}\frac{l!\ \exp(i(l+1)\varphi)}{j!\left(\rho^2\sigma_E\right)^j}r^{2j-l-1}\right\}\times\exp\left(-\frac{r^2}{\rho^2\sigma_E}\right), \tag{22}$$

$$\left|0 \quad l+1\right\rangle^{+}_{TM} == B_M\left\{\hat{\mathbf{c}}^+\frac{r^{l-1}\exp(i(l-1)\varphi)}{\left(\rho^2\sigma_M\right)^l}-\hat{\mathbf{c}}^-\sum_{j=0}^{l}\frac{l!\ \exp((l+1)\varphi)}{j!\left(\rho^2\sigma_M\right)^j}r^{2j-l-1}\right\}\times\exp\left(-\frac{r^2}{\rho^2\sigma_M}\right), \tag{23}$$

$$\left|0 \quad l+1\right\rangle_{TE}^{-} == B_E \left\{ \hat{\mathbf{c}}^{+} \sum_{j=0}^{l} \frac{l!\ \exp(-i(l+1)\varphi)}{j!\left(\rho^2 \sigma_E\right)^j} r^{2j-l-1} + \hat{\mathbf{c}}^{-} \frac{r^{l-1} \exp(-i(l-1)\varphi)}{\left(\rho^2 \sigma_E\right)^l} \right\} \times \exp\left(-\frac{r^2}{\rho^2 \sigma_E}\right), \quad (24)$$

$$\left|0 \quad l+1\right\rangle_{TM}^{-} = B_M \left\{ \hat{\mathbf{c}}^{+} \sum_{j=0}^{l} \frac{l!\ \exp(-i(l+1)\varphi)}{j!\left(\rho^2 \sigma_M\right)^j} \times r^{2j-l-1} - \hat{\mathbf{c}}^{-} \frac{r^{l-1} \exp(-i(l-1)\varphi)}{\left(\rho^2 \sigma_M\right)^l} \right\} \times \exp\left(-\frac{r^2}{\rho^2 \sigma_E}\right), \quad (25)$$

where $B_E = (-1)^{l-1} z_{0E} 2^l \quad B_M = (-1)^{l-1} z_{0M} 2^l$. Now the set of the Laguerre-Gaussian beams represented by eqs (16)-(19) and (22)-(25) is a full one and any light beam at the z=0 plane can be expressed by means of the superposition of these equations. It should be noted that this superposition enables us to get rid of the amplitude uncertainty sprung up in eqs (22)-(25) at the axis $r = 0$.

Involves attention the fact that the expression

$$\begin{aligned} |\Psi\rangle \propto \hat{\mathbf{c}}^{+} \frac{1}{\rho^2} &\left(\frac{e^{-\frac{r^2}{\rho^2 \sigma_e}}}{\sigma_e} + \frac{e^{-\frac{r^2}{\rho^2 \sigma_m}}}{\sigma_m} \right) - \\ -\hat{\mathbf{c}}^{-} e^{i2\varphi} &\left[\frac{e^{-\frac{r^2}{\rho^2 \sigma_e}} - e^{-\frac{r^2}{\rho^2 \sigma_m}}}{r^2} + \right. \\ &\left. +\frac{1}{\rho^2} \left(\frac{e^{-\frac{r^2}{\rho^2 \sigma_e}}}{\sigma_e} - \frac{e^{-\frac{r^2}{\rho^2 \sigma_m}}}{\sigma_m} \right) \right] \end{aligned} \quad (26)$$

obtained as the superposition eqs (22) and (24) coincides with the identical expression in the article [3] obtained by the spectral integral technique. The wave function in eq. (26) reflects the evolution of the initial fundamental Gaussian beam: the topological structure of the beam component with initial circular polarization $\hat{\mathbf{c}}^{+}$ does not undergo any transformations while there is the phase singularity in the form of the optical vortex with the double topological charge nested in the beam with the $\hat{\mathbf{c}}^{-}$-polarization component.

On the other hand, the problem of the beam transformation in the crystal can be approximately solved by other way as in article [2]. Indeed, a Gaussian beam can be represented as a superposition of light rays localized on the surface of hyperboloids. Each a ray is associated with a plane wave. The wave propagation through the crystal is described on the base of Jones's matrix formalism. As a result, we obtain the approximate expressions that are in a good agreement with the experiment.

Consider some conformity between the exact theory presented above and the approximate results obtained in [2]. As far as the condition $\alpha = \Delta\varepsilon / \varepsilon << 1$ takes place in a real crystal then

$$\begin{aligned} \frac{1}{\sigma_E} - \frac{1}{\sigma_M} &\approx i \frac{z}{z_{0E}} \frac{\alpha}{\tilde{\sigma}^2} \\ \frac{1}{\sigma_E} + \frac{1}{\sigma_M} &\approx \frac{2}{\tilde{\sigma}} \end{aligned}, \quad (27)$$

where $\tilde{\sigma} = \sigma_E + i \frac{\alpha}{2} \frac{z}{z_{0E}}$. Using now the superposition $\mathbf{\Psi}_{0,l} = \left|0 \quad l+1\right\rangle_{TM} - i \left|0 \quad l+1\right\rangle_{TE}$, eqs (22) and (23) we find

$$\begin{aligned} \mathbf{\Psi}_{0,l} = &\propto [r/\sigma]^{l-1} \left[\hat{\mathbf{c}}^{+} \cos(\delta/2) + \right. \\ &\left. + \hat{\mathbf{c}}^{-} i \sin(\delta/2) \exp(i2\varphi)\right] \times \\ &\times \exp\left(-\frac{r^2}{\rho^2 \tilde{\sigma}}\right) \exp(i(l-1)\varphi) \end{aligned}, \quad (28)$$

where $\delta \approx k \Delta n r^2 / z$ provided that $\Delta\varepsilon \approx 2 n_e \Delta n$ and $\Delta\varepsilon \approx 2 n_e \Delta n$. The last condition corresponds to a real light focusing in crystals. The equation (28) coincides with the identical

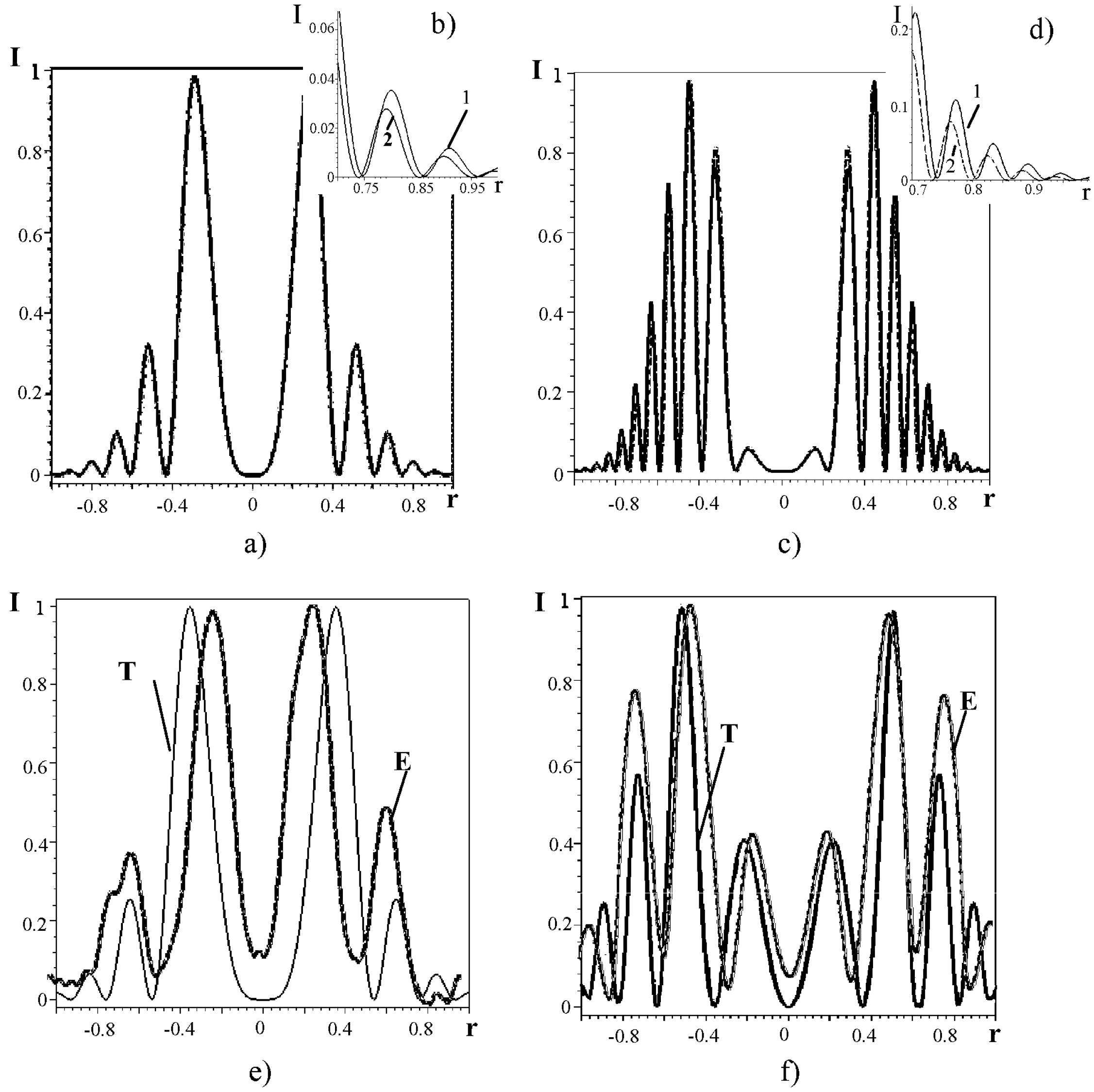

Fig.1. The theoretical and experimental intensity distributions of the high-order mode beams: (a,e) – n=m=0, $\mathbf{c}^{+}$ - component; (b) – the fine structure of the curve in Fig.1a; (c) – n=m=1, $\mathbf{c}^{-}$ - component; (d) the fine structure of the curve in Fig.1c; (curve 1 – the exact theory, curve 2 –the approximate theory) (f) n=m=1, $\mathbf{c}^{+}$ - component (T – the approximate theory, E- the experiment).

approximate expression in the work [2] up to the constant factor.

Two set of the curves – 1 and 2 presented in Fig.1(a-d) illustrate the dependency of the beam intensity I on the radial coordinate r. The curves 1 are calculated in accordance with the expression (26) while the approximate eq. (28) is associated with the curves 2. The curves in Fig.1(e,f) point out a good agreement between the theory and experiment in the work [2]. At the same time, a good agreement of the approximate and exact theory is broken for the beams with indices $n > 1$.

Thus, starting from the vector wave equation we have obtained exaxt expressions for the full set of the Laguerre-Gaussian beams with a complex argument inside a uniaxial crystal. We have discussed the limits of applicability of the approximate theory worked out in the article [2] and thereby showed that Jones's matrix formalism can be used for description of the beam propagation in a uniaxial crystal. At the same time, the matrix formalism in the given form cannot be expanded on the Laguerre-Gaussian beams with the $n > 1$ radial indices.

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