Dimensioning of Perfect and Regularly Transformed Optical Fractals

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Abstract

The peculiarities of dimensioning regular fractals through measuring power spectra of the diffracted optical radiation field are discussed. It is shown on the example of one-dimensional Cantor set that asymmetrization of the fractal structure leads to modifications of the corresponding Fraunhofer diffractal, which nevertheless preserves its global symmetry. For that, the slope of the amplitude spectrum represented in a log-log scale is invariant in respect to the asymmetry coefficient applied to the fractal structure, being directly associated with the fractal dimension of the object.

Keywords: optical fractals, diffraction, fractal dimension, power law, amplitude spectrum

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1. Introduction

Optical techniques for investigation of the fractals, i.e. objects possessing structural self-similarity under scale changing and non-integer dimension, are attractive owing to high operation speed and their non-contact (non-destructive) nature [1]. Thus, the main diagnostic parameter such as fractal dimension can be estimated from the slope of a power spectrum of the diffracted radiation field represented in log-log scale and known as a *power law* [2]. Optical dimensioning of a fractal object is reduced to Fourier transformation of the object's boundary field (by a free-space layer of sufficient extent or, more often, using a Fourier-transform lens [3]), and measuring of spatial intensity distribution (equivalently: power spectrum or the squared complex Fourier-spectrum modulus) at Fraunhofer diffraction region.

Numerous investigations in the field of Fractal Optics concerning artificially prepared deterministic (perfect) fractals [1,2,4] whose properties are known in advance being pre-determined by the construction algorithm. On the other hand, most of real (natural) fractals are to a certain extent, randomized and asymmetrical being at the same time characterized by a complicated structure of the edges of structural elements. It has been found [1] that the direct dimensioning of the fractal objects on the power law is possible in the case of random fractals only. In contrast, dimensioning of a regular fractal through intensity distribution at the Fraunhofer diffraction region is problematic. To our knowledge, the cause of the mentioned problem has not been elucidated up to now.

Our previous investigation of the optical fields produced by diffraction of a probing radiation at the fractal objects (kind of Koch curves transformed by smoothing or enhancing of the structural element edges) show [5] that such fields possess the properties connected with a self-similarity of the structures of interest. Besides, in the paper [6] we studied one-dimensional Cantor sets which had undergone transformation by displacement of the edges (and, as a consequence, the «centers of mass») of the structural elements, i.e. transparent bars, for the specified magnitudes of the asymmetry coefficient. It has been shown, in part, that in the near (Fresnel) diffraction region the magnitude of the asymmetry coefficient of the diffracted field intensity distribution is close to the magnitude of this coefficient at the boundary object field, whereas in the far (Fraunhofer) diffraction region the magnitude of the asymmetry coefficient approaches zero; in other words, Fraunhofer diffractal is symmetrizied. It means that a power law is inapplicable for determining the object's asymmetry. At the same time, another question is left without an adequate answer: is the object's fractality reflected in a power law of the asymmetrizied fractal? In other words: can one determine the object's fractal dimension using a power law irrespective of the object asymmetry? A theoretical solution of this problem meets with considerable difficulties both of computational nature and interpretation. In particular, there is no common opinion as to considering an asymmetrical fractal as the monofractal, which is completely characterized by the unique non-integer dimension, or as the multifractal, which is characterized by a set of non-integer dimensions or so-called spectrum of singularities [4].

In this paper we describe the procedure of dimensioning of perfect and regularly transformed one-dimensional Cantor sets from experimentally obtained power law and discuss the results using an analogy of a Cantor set with a diffraction grating.

2. Experimental procedure

The Cantor set of the third level is used as the object. The perfect (symmetrical) Cantor pre-fractal is asymmetrizied using the algorithm introduced in Ref.6. Namely, the coordinates y of the transformed fractal correspond to the coordinates x of the perfect fractal by the rule $y = x^{1+k}$, where k is the asymmetry coefficient ($k = 0$ corresponds to the identity transformation). Note that the maximal asymmetry coefficients realized in our experiment exceed those used in [6] approximately by one order of magnitude: $k = 1.5$ against $k = 0.2$. Thus, strongly deformed fractals, are studied.

The experimental samples with the desirable magnitudes of an asymmetry coefficient shown in figure 1 are prepared in the following manner. The perfect and the transformed fractals are obtained by means of computer simulation and printed on *A4* format paper using a laser printer. The obtained amplitude fractals are photocopied using a low-contrast film *ORWO*, whose contrast coefficient $\gamma \approx 1$, and resolving power (up to 300 lines/mm) provides perfect reproduction of the edges of the fractal structural elements. Thus, an amplitude transmittance of the film samples directly corresponds to the amplitude Cantor sets. The width and the height of the fractals used in our experiment are 5.7 mm and 27 mm, respectively.

An optical set-up for recording the Fraunhofer diffractals shown in figure 2 is the same as the set-up realized in Ref.2. The beam of He-Ne laser radiation ($\lambda = 0.6328\,\mu$m, power $P = 50\,$mW) is focused by the microscopic objective ($20^{\times}$) and filtered by a pinhole of $28\,\mu$m – diam. The quasi-point source, *PS*, formed in such a manner is transformed into the quasi-linear source, *LS*, using a cylindrical lens *CL*. A quasi-linear source is imaged by the objective *O* ($f = 40\,$mm) into the plane *RP*. The fractal *F* is placed between the objective *O* and the plane *RP* and uniformly illuminated by the converging wave whose cross-section greatly exceeds the

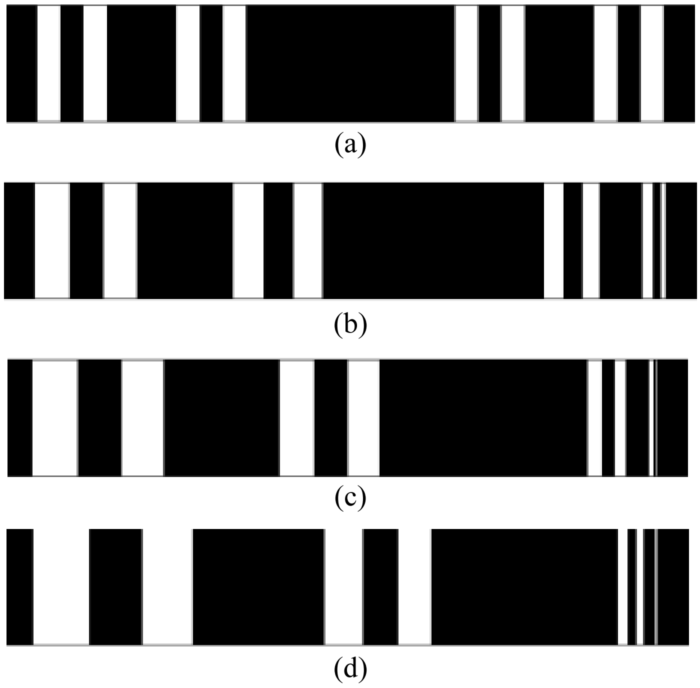

Fig. 1. Experimental samples: the perfect - *k=0* (a), and the asymmetric - *k=0.5* (b), *k=1* (c), and *k=1.5* (d) Cantor sets.

width of the input fractal aperture. It is well known [3], that such positioning of the object possesses important advances:

- the probing beam is modulated by the input aperture along one spatial coordinate only;
- aperture limitations along the bars of the

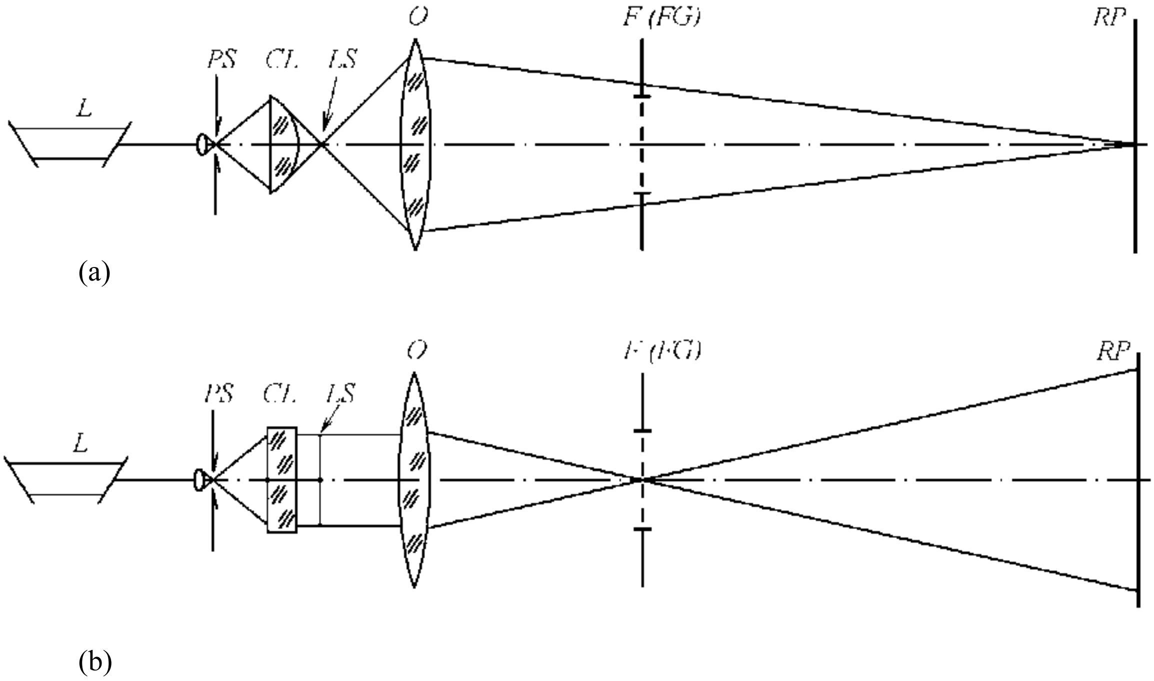

Fig. 2. Arrangement for a hologram recording: its top view (a) and side view (b). *L* - He-Ne laser, *PS* - quasi-point source (microscopic objective and a diaphragm as the spatial filter), *CL* - cylindrical lens, *LS* - quasi-linear source, *O* - objective, *F (FG)* - fractal (or fractalogram), *RP* - registration plane.

Cantor set are absent: the experiment is implemented with a true one-dimensional fractal;
- power of the probing radiation is used in the most efficient manner;
- the effects of spatial-frequency filtering are avoided;
- the scale of a power spectrum and autocorrelation image of the input signal is controlled in the simplest manner.

A power spectrum (actually, Fraunhofer diffractal that is equal to the referenceless Fraunhofer frac-talogram [7]) is recorded at Russian amplitude holographic photoplate VRL developed in developer D-19. Note that in this experiment we provide «soft» (relatively cold and long-duration) developing. It permits to avoid the influence of the effects of nonlinear holographic recording [8] on the results of measurement.

Further, replacing a fractal *F* (see figure 2) by its fractalogram *FG*, we obtain an autocorrelation image of the studied fractal at the plane *RP* as the Fourier transform from a power spectrum, in agreement with the Wiener-Khintchine theorem. The distances from the objective *O* to the plane *RP* and from the input aperture to the same plane are $L_0 = 1.32 \cdot 10^3$ mm and $L_1 = 10^3$ mm, respectively.

The set-up shown in figure 2 is also used for measuring the intensity distribution within the central (self-similar [3]) areas of the corresponding power spectrum and autocorrelation image of the perfect fractal and the transformed ones. In this case, a holographic photoplate at the plane *RP* is replaced by a photodetector (in our experiment by a photomultiplier, *PMP*). The *PMP* is mounted on the cross-head, which is rotated about the origin of coordinates of the input plane and moves perpendicularly to the main optical axis of a coherent-optical processor as well as to the Cantor bars. A slit diaphragm of the width $\sim 100\,\mu$m is placed just before a *PMP*. Much exceeding (by 160 times, approximately) the wavelength of the probing radiation, such a diaphragm does not cause any polarization effects and, consequently, does not influence the measurement results. Its role is reduced to diminishing the noise and providing desirable angular resolution when the actual intensity distribution is measured. Under the wavelength and geometrical conditions of our experiment, a slit width is twice narrowed in respect to the minimal period of the studied diffraction pattern. For that, an angular uncertainty associated with the finite (non-vanishing) width of the analyzed diaphragm is 10^{-4} rad, and the angular uncertainty associated with the finite width of the primary source *PS* is $L_0 = 1.6 \cdot 10^{-4}$ rad. Thus, the resulting uncertainty of the diffraction angle is $\delta\theta \approx 2.6 \cdot 10^{-4}$ rad ($\le 1.5 \cdot 10^{-2}$ deg) that corresponds to the spectral resolution $\Delta \cong \delta\theta/\lambda \sim 4.1 \cdot 10^{-1}\,\text{mm}^{-1}$. The last estimation characterizes resolution of two wave vectors associated with the corresponding components of the Fourier expansion of the input signal realized in our experiment, being equal to the best parameters presented in literature [2,4].

The width of the analyzed area is the same for all studied fractals. Its warranty covers the central (self-similar) domain whose extension is determined by the fractal level. As all the studied objects are of the same level, the maximal diffraction angle where intensity of the field is measured is $1.5 \cdot 10^{-2}$ rad, which provides separate measuring up to 57 intensity magnitudes for the mentioned $\delta\theta$. The distinguishing feature of optical power spectra and autocorrelation images of the Cantor set consists of a large modulation percentage of the spatial intensity distribution. That is why we use the calibrated neutral attenuators just in front of the analyzing slit to provide measurement linearity over all range of the measured intensity magnitudes. The number and multiplicity of the used attenuators are taken into account under experimental data processing.

3. Qualitative visual analysis

The experimentally obtained power spectra of the perfect fractal and the transformed ones, i.e. referenceless Fraunhofer fractalograms, are shown in figure 3. A power spectrum of the perfect fractal (Figure 3(a)) may be conventionally divided into two domain: one of them is the central («inner») domain of relatively high intensity possessing a structural self-similarity, and the other is the peripheral («outer») domain of relatively low intensity possessing a periodical structure. Indeed, three clearly separated sets of spectral components are observed within the «inner» domain, and each set can be divided into three parts consisting, in turn, of three lines. As it is well known [2], finiteness of the self-similar domain is caused by the finite level of the corresponding pre-fractal. The obvious triadic structure of a self-similar domain of a power spectrum is the direct consequence of the constructing algorithm for a Cantor set.

Power spectra of all transformed fractals (Figure 3(b)-(d)) have complicated intensity distributions in comparison with a power spectrum of the perfect fractal. As a result, it is difficult to divide such spectra in to the above mentioned typical domains. The number of spectral components grows, and intensity distribution at the diffraction pattern is smoothed out. The presence of the common spectral component corresponding to the first diffraction maximum in the periodical domain of power spectrum of the perfect fractal is explained as the consequence of equal width of all studied samples. As the magnitude of an asymmetry coefficient of the transformed fractals grows, the width of the central domain grows also which follows from the diminishing of the minimal width of some structural

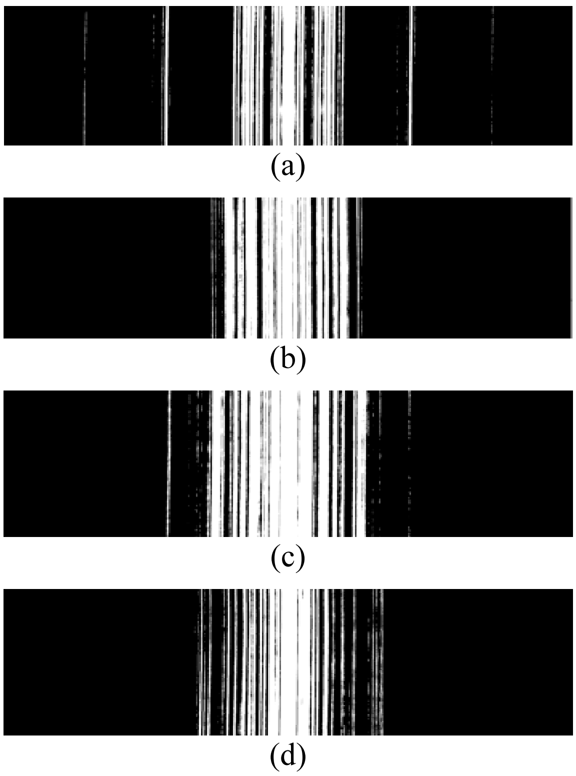

Fig. 3. Power spectra of the perfect (a) and transformed (b) to (d) Cantor sets. Asymmetry coefficient magnitudes are the same as in the corresponding fragments of figure 1.

elements of the input aperture.

Autocorrelation images of the studied fractals are shown in figure 4. An autocorrelation image of the perfect fractal (Figure 4(a)) is characterized by the existence of clearly separated diffraction maxima within the periodical domain without any observable features of structural self-similarity within the central domain. Similarly to the corresponding power spectra, autocorrelation images of the asymmetrizied fractals (Figure 4(b)-(d)) are of complicated structure being characterized by more rapid decreasing of intensity with increase of spatial frequency in comparison with an autocorrelation image of the perfect fractal. Triadic structure of the fractal is not reflected directly in its autocorrelation image irrespectively to the magnitude of the asymmetry coefficient. In contrast to the corresponding power spectrum, the width of the central domain of an autocorrelation image decreases as the magnitude of the asymmetry coefficient increases. Thus naturally follows from the method of obtaining an autocorrelation image as well as from the well known properties of optical Fourier transform [3]. Note that all power spectra and autocorrelation images shown in figures 3 and 4 are globally symmetrical irrespective of the magnitude of the asymmetry coefficient characterizing the input signal.

4. Measurement results

The measured spectra of the perfect and asymmetrizied Cantor sets are shown in log-log scale in figure 5. The magnitudes of amplitude (left scale) and intensity (right scale) of the diffraction maxima normalized on the forward-diffracted wave amplitude and intensity, respectively, are shown by the spots. The experimental spots in the two mentioned representations coincide with each other owing to the matched scale transforming (square rooting) of the imaged dependence. The solid and the dashed straight lines are the approximating dependencies $\sqrt{I(q)} \sim q^{-D}$ and $I(q) \sim q^{-D}$, respectively, where q is the spatial frequency. For the triadic Cantor set

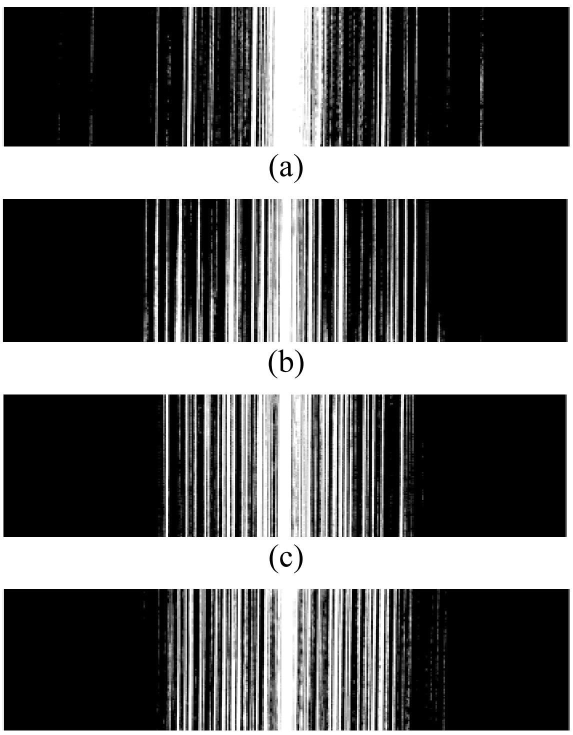
(a) (b) (c) (d)

Fig. 4. Autocorrelation images of the perfect (a) and transformed (b) to (d) Cantor sets. Asymmetry coefficient magnitudes are the same as in the corresponding fragments of figure 1.

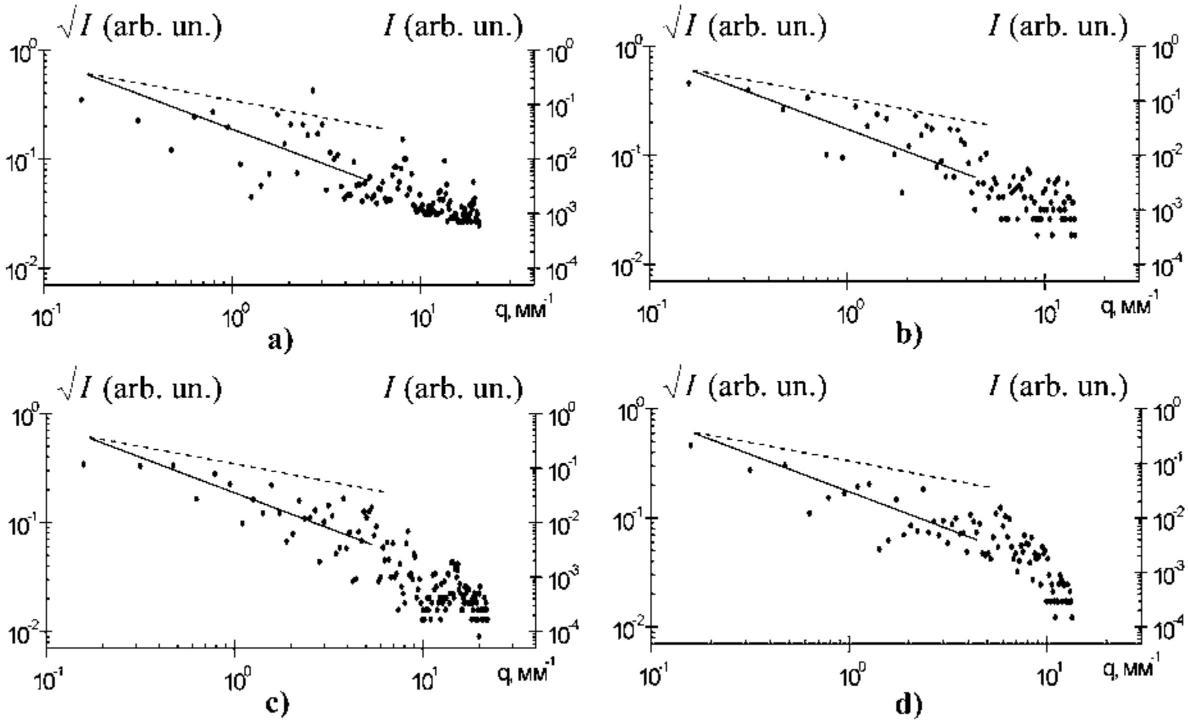

Fig. 5. Experimentally found dependencies $\log\sqrt{I}(\log q)$ and $\log I(\log q)$ for spectra of the perfect (a) and asymmetrical (b) to (d) fractals. Solid and dashed curves correspond to the slope of *D* in scales $\sqrt{I}$ and I, respectively.

$D = \ln 2/\ln 3 \approx 0.6309$ [2]. Both dependencies are presented for the central (self-similar) domain of the Fraunhofer diffractal.

The conclusions following from the results shown in figure 5 are:

1. The slope of the dependence $\log\sqrt{I}(\log q)$ directly corresponds to the theoretically predicted dimension for the perfect fractal: the number of experimental spots above the solid line is the same as the number of spots under this line. This conclusion has been also verified by applying a least squares approximation technique. For that, the discrepancy between the slope of the theoretical dependence and the optimal approximation of experimental data does not exceed 5%.

2. The dependence $\log I(\log q)$ is not applicable for direct dimensioning of the perfect fractal in agreement with the conclusions of Ref.2. It has been found that the dependence of intensity of the Fraunhofer diffractal on spatial frequency is fitted by the interrelation $I(q) \sim q^{-(2-D)}$ within the error inherent to the approximation $\sqrt{I(q)} \sim q^{-D}$ (5%).

3. Approximation $\sqrt{I(q)} \sim q^{-D}$ is applicable for the same extent for dimensioning both the perfect fractals and the asymmetrical ones, as it is seen from figure 5. It is remarkable that the reliability of such approximation even increases when the magnitude of the asymmetry coefficient grows.

4. Outside the limits of self-similar domain of the Fraunhofer the diffractal trivial dependence [1] $I(q) \sim q^{-1}$ is realized (it is not shown in figure 5).

The autocorrelation images of the perfect and asymmetrizied fractals are obtained through Fourier transformation of the corresponding power spectra. The experimentally found amplitude and intensity distributions into autocorrelation images are shown in log-log

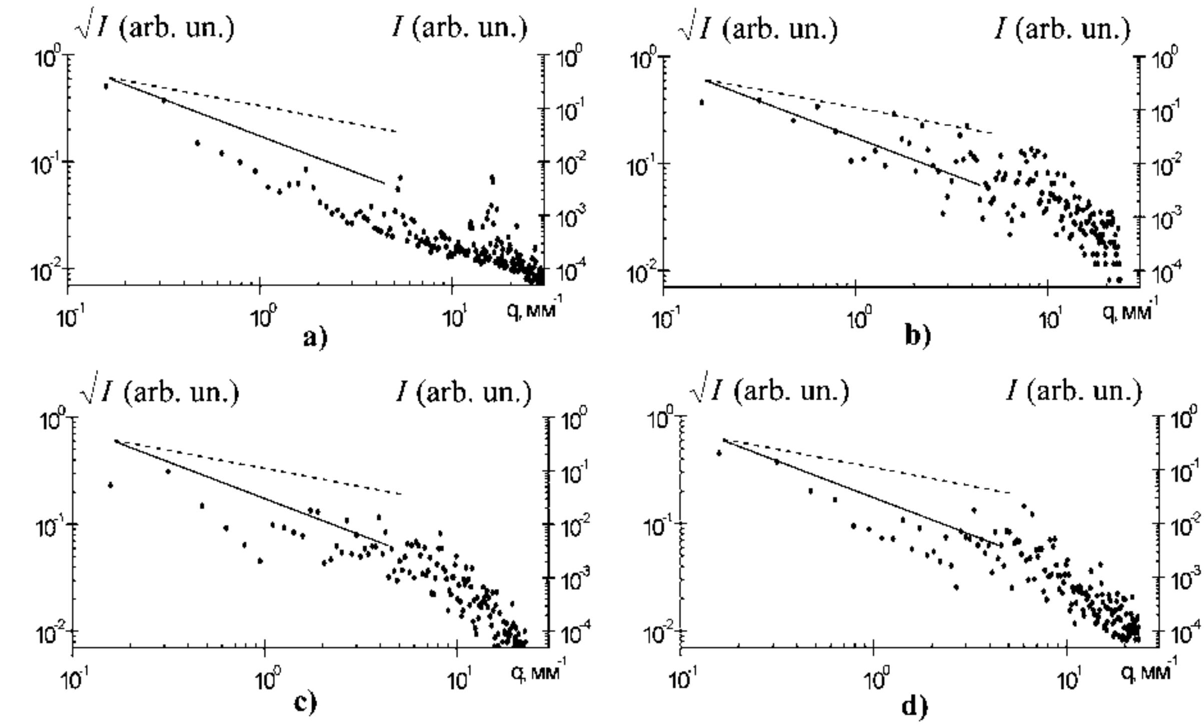

Fig. 6. Experimentally found dependencies $\log\sqrt{I}(\log q)$ and $\log I(\log q)$ for autocorrelation functions of the perfect (a) and asymmetrical (b) to (d) fractals. Solid and dashed curves correspond to the slope of *D* in scales $\sqrt{I}$ and I, respectively.

scale in figure 6. The slope of the dependencies of intensity distributions on spatial frequency within the central domain is close to the slope of the dashed line, $\sim(2-D)$, and the slope of the dependencies of the amplitude distribution on spatial frequency within this domain turns out to be close to D (solid line). However, the discrepancy between the slope of the optimal approximation of the amplitude distribution and dimension D occurs considerably larger both for the perfect fractal and the asymmetrizied ones than the discrepancy between the optimal approximation of the experimental data and the dependence q^{-D} for the corresponding spectra, being now within the limits from 15% (for $k=0.5$) to 30% (for $k=1.5$).

It follows from the presented results that deterministic fractals (both simmetrical and asymmetrical) can be dimensioned from the slope of the dependence $\log\sqrt{I}(\log q)$, whereas reconstruction of autocorrelation images of the fractals through optical Fourier transformation of the amplitude transmittance of referenceless Fraunhofer fractalograms turns out to be redundant being neither necessary nor efficient means for the dimensioning of fractal objects.

5. Discussion

In paper [2], one of the initial studies devoted to fractal objects dimensioning through measurements at the field of radiation diffracted by such objects, it is established that fractal dimension may be found on the slope of the envelope of the amplitude Fourier spectrum, i.e. the dependence of Fourier-amplitude of the field on spatial frequency represented in the log-log scale. But the experimental results [2], in part for the Cantor set, are shown for another dependence, namely for the intensity of the Fraunhofer diffractal against spatial frequency. The approximation of the measured power spectrum by the dependence q^{-D} represented in Ref.2 is far from the optimal one. Our attempt to find the approximating dependence for experimental data [2] has shown that the fitting curve is a parabola rather than a straight line as it is predicted by the theory derived in Ref.2. At the same time, the approach proposed in Ref.2 occurs to be quite adequate being applied to random fractals, in part to so-called diffusion-limited aggregates [4,9]. Besides, it follows from the results of Ref.10 that randomization of the fractal structure leads to a decreasing of the discrepancy among the theoretical and experimental results on fractal dimensioning. Let us note that various averaging techniques discussed in [1] are not substantionally grounded being in essence only empirical fitting procedures, and, moreover, are not applicable to one-dimensional fractal kind of Cantor sets studied in our paper. From the paper [1], it is regarded as agreed-upon that the procedure of the estimation of fractal dimension on the power law is efficient only for random fractals.

Inapplicability of a power law for estimating D is also illustrated by our results (see dashed lines in figure 5). But at the same time we show the possibility for direct dimensioning of regular fractals on the slope of the dependence of the amplitude Fourier spectrum. Let us explain here the considerations, which led us to the necessity of taking the square root of the power spectrum of radiation diffracted by the perfect (or regularly transformed) fractal for the dimensioning of fractals on the optical experiment data. The reasons expounded below are close to those presented in [11] where the difference between autocorrelation images of deterministic input signals (kind of a diffraction aperture) and diffuse ones is considered.

The perfect Cantor set is none other than the regular grating with the «omitted» (on specific rule) transparent elements. If so, the use of the theory of diffraction gratings seems to be the simplest and the most convincing way for discussing the above mentioned problem.

The origin of any main diffraction maximum results from an in-phase addition (in the corresponding direction or in the corresponding point) of partial cylindrical waves propagating from various transparent elements of the grating. The resulting field amplitude, A, for the specified conditions of observation is the sum of the partial amplitudes: $A \sim N$, where N is the number of grating periods; thus, the resulting field intensity is $I = A^2 \sim N^2$.

Let us now consider the result of grating randomization. Such a randomization may be implemented in two ways:

- phase randomization of the boundary object field due to usage of a diffused;
- random displacements of the edges (as a consequence, random displacements of the «centers of mass» of the grating structural elements), i.e. deformation of the grating elements.

The last case is actual within the context of our study. In accordance with the shift theorem of Fourier analysis [6], the directions on the main diffraction maxima are unchanged, and the shift of the signal at the spatial domain causes the corresponding linear (in the case under consideration, random) phase shift at the frequency domain. In the limiting case when the phase shifts are distributed uniformly within the interval from 0 to 2π, the resulting intensity of the main diffraction maximum occurs (in agreement with Ref.12, section 42*a*) to be equal to the number of partial pixels of the input signal, $I \sim N$. Thus, *intensity* of the main diffraction maximum of the randomized grating (including the fractal one) obeys the same law as *amplitude* of the corresponding maximum in the case of regular grating. This conclusion highlights the fact that the envelope of the intensity frequency distribution at the randomized fractal Fourier-transformation coincides with the envelope of the amplitude frequency amplitude distribution of the Fourier-transformation of the deterministic (symmetrical or asymmetrizied, i.e. regularly transformed) fractal. Indeed

- the conclusion of Ref.2 regarding the

possibility of fractal object dimensioning on Fourier-spectrum of the diffracting field is justified in practice when the magnitude of D is estimated directly on the measured intensity distribution at a far field of a random fractal;

- estimating the magnitude of D for regular fractal is mediated by the procedure of finding the amplitude Fourier-spectrum of the object field in its power spectrum.

Let us note that at considerable (by N-times) decreasing of intensity at the main diffraction maxima, which accompanies randomization of a regular fractal, the total energy of the diffracted radiation is unchanged. This can be easily shown within the framework of the Young-Rubinowicz model of diffraction phenomena [13-15] where the diffraction field is computed as the sum of the geometrical optics wave and the edge diffraction wave. The geometrical optics wave is defined within the directly illuminated area only, i.e. within the central diffraction maximum of a far-field diffraction pattern [7]. An increase of transmittance of the asymmetrizied fractal (due to the increase of a whole area of the transparent elements (see figure 1), leads to the increase of the intensity of the central diffraction maximum only. At the same time, the number of the structural elements of the transformed fractals is the same as one in the initial regular fractal. Thus, the total energy of the partial edge diffraction waves is conserved.

It is clear that the decrease of intensity of the main diffraction maxima is accompanied by the increase of a diffuse component. Namely, now the light comes to the areas of a far-field pattern, which earlier were «dark» ones. Of course, the partial signals are added in intensity within these areas. The structure of the transformed Fraunhofer diffractal is enriched with the new elements that does not affect the structure of the spectrum envelope. The last conclusion is illustrated, in part (for the case of the regularly transformed perfect fractal), in figures 3 and 5, being quite in agreement with the results [10] showing that randomization of two-dimensional Koch fractal causes the evolution of self-similar Fraunhofer diffractal to the random speckle-pattern.

6. Conclusions

The following conclusions follow from the above described study.

1. Dimensioning of the determinate regular fractals through measurement of the angular intensity distribution of radiation diffracted by such fractals is possible and presumes the *amplitude spectrum* of the scattered radiation to be found, the envelope slope of whom (represented in log-log scale) corresponds to the dimension of the object.

Even for considerable magnitudes of an asymmetry coefficient (up to $k = 1.5$), the slope of an optimal approximation of experimental data differs from the theoretically predicted fractal dimension less than by 5% that approximately coincides with the measurement error.

2. As the fractal is asymmetrizied, the observed features of structural self-similarity of the object in a far-field diffraction pattern are gradually smoothed. Nevertheless, the slope of the envelope of the dependence $\log\sqrt{I}(\log q)$ found from the measured intensity at Fraunhofer diffractal occurs to be invariant against changes of the asymmetry coefficient of the object. This slope can be used as the diagnostic parameter. For that, the described procedure is sufficient both for revealing the object fractality of the object and for dimensioning asymmetric fractals, being at the same time inadequate for estimating the coefficient of asymmetry due to global symmetry of a far-field diffraction pattern.

3. At last, the results of our study show that the asymmetrizied Cantor fractal remains a mono-fractal characterized by unique fractal dimension.

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