Optical Superposition of the Modulated Images in the Fractional Fourier Transform Domain

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Abstract

The general theory of forming the fractional Fouier transform images of two shifted and the plane wave modulated optical signals is given. Using this theory, we have shown that the parallel process of both cross shifting and modulation of the image by plane wave take place, if the fractional Fouier transform is realized. The principal possibility of the fractional Fouier transform images optical superposition for an arbitrary value of the fractional Fouier transform parameter p is determined. The influence of the interference term on forming the FFT conjugate images is shown. Results of numerical calculations of two optical superimposed fractional Fouier transform images of rectangular impulses are presented.

Keywords: fractional Fourier transform, fractional correlation, signal distribution method, rectangular slit, image forming, cross shifting.

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1. Introduction

Sufficient progress in the fractional Fourier transform (FFT) theory development [1] stipulates intense investigations on the creation of a principally new optical correlator, which can be used for pattern recognition and image processing.

It is known, that the FFT of optical signal $f(x)$ may be written in the form of integral transform [1,2]

$$u_p(x) = \hat{\Im}^p[f(x)] = C_0 \int_{-\infty}^{\infty} f(x_0) \exp\left(i\frac{k[x_0^2 + x^2]}{2d_0 \operatorname{tg}\phi} \right) \times \exp\left(-i\frac{kx x_0}{d_0 \sin\phi} \right) dx_0, \quad (1)$$

where $\hat{\Im}^p$ is an FFT operator, C_0 is a constant factor:

$$C_0 = \sqrt{\frac{k}{2\pi d_0}} \frac{\exp(i[\pi/4 - \phi/2])}{\sqrt{\sin\phi}}, \quad (2)$$

$k = 2\pi/\lambda$ is a wave number, d_0 is a linear constant, $\phi = p\pi/2$, p is an FFT parameter.

A conjugate FFT $U_p(x) = \hat{\Im}[u_p(x)]$ of an optical signal $f(x)$ in common with $u_p(x)$ was proposed in [2,12]. Comparative analysis of couple of conjugate FFT $u_p(x)$ and $U_p(x)$ related by a Fourier transform operator allow to discover the common regularities and pecularities of a FFT process in full measure.

A classical scheme of a joint transform correlator (JTC) [3] consists by the introduction of two shifted optical signals $g(x, y) = f_1(x+b, y) + f_2(x-b, y)$ into the correlator input plane (x, y). The joint power spectrum of optical signals is produced in the spatial domain (ω_x, ω_y) of the correlator and is registered as

$$|G(\omega_x,\omega_y)|^2 = |F_1(\omega_x,\omega_y)|^2 + |F_2(\omega_x,\omega_y)|^2 + F_1(\omega_x,\omega_y)F_2^*(\omega_x,\omega_y)\exp(2ib\omega_x) + F_1^*(\omega_x,\omega_y)F_2(\omega_x,\omega_y)\exp(-2ib\omega_x), \quad (3)$$

where $F(\omega_x, \omega_y)$ is a Fourier image of an optical signal. In the second cascade of a JTC, the correlation function of two signals is optically realized as a result of inverse Fourier transform:

$$C_{12}(x,y) = \hat{\Im}^{-1}\left[F_1(\omega_x,\omega_y)F_2^*(\omega_x,\omega_y)\right]. \quad (4)$$

A number of papers [4-6], offer considering the general scheme of an optical signal correlation in the FFT domain as:

$$C_{12}^{(p,q,r)}(x) = \hat{\Im}^p\left[\hat{\Im}^q\left[f_1(x)\right]\hat{\Im}^{r^*}\left[f_2(x)\right]\right] \quad (5)$$

If $p=-1; q=r=1$, we obtain a Fourier transform correlator scheme as a particular case.

The different cases of fractional correlation depending on the FFT parameter were investigated in [7]. It was shown that such schemes of a generalized correlator are less sensitive to additive noises.

The principle of operation of a JTC is constructed on Fourier analysis methodology of two mutually shifted input signals. To find principally new optical schemes of a generalized correlator, it is important to investigate the properties of mutually shifted signals in the process of the FFT realization.

It is known that FFT of optical signals may be characterized by two important properties:
cross shifting of the signal on the value b

$$\hat{\Im}^p[f(x-b)] = \exp\left(i\frac{b^2}{2}\frac{k}{d_0}\sin\phi\cos\phi\right) \times \exp\left(-ib\frac{k}{d_0}\sin\phi\, x\right)u_p(x-b\cos\phi). \quad (6)$$

signal modulation by the plane wave with a frequency ω_1

$$\hat{\Im}^p[f(x)\exp(-i\omega_1 x)] = \exp\left(-i\frac{\omega_1^2}{2}\frac{d_0}{k}\sin\phi\cos\phi\right) \times \exp\left(-i\omega_1\cos\phi\, x\right)u_p\left(x+\omega_1\frac{d_0}{k}\sin\phi\right). \quad (7)$$

Thus, the cross shifting of an optical signal leads to cross shifting of the FTT proportionally to $\cos\phi$ and to modulation (up to a constant quadratic phase factor) by a harmonic signal with a frequency proportional to $\sin\phi$. This property is of fundamental importance. Alieva [4] considered the shift of the FFT for different values of FFT parameter in detail. This property was also considered in the paper [8] in the case of two signals, one of which is shifted and modulated by the plane wave with frequency ω_1.

In [3, 9-14], the approach to an analytical description of the FFT properties based on using the coordinate-frequency method of signals [10, 11] is formulated. In particular, it is shown that the FFT causes the rotation of an input signal proportionally to parameter p. Besides, the FFT image is restored by the inverse Fourier transform operator $\hat{\Im}^{-1}$. A conception of the FFT conjugate image forming is formulated in [12]. It is shown by numerical results [2, 13, 14] the redistribution of conjugate images $|u_p(x)|^2$ and $|U_p(x)|^2$ occurs with a continuous changing of a parameter p in two directions from a coordinate to a spatial planes and vice versa. In the point $p=1/2$, the conjugate images $|u_p(x)|^2$ and $|U_p(x)|^2$ are degenerated in the self-similar for an arbitrary optical signal.

In this work, the theory of the FFT image forming is given for a general case of two shifted and plane modulated wave optical signals. The analytical calculation of the FFT conjugate images is considered and the theoretical basing of a principal possibility of the FFT image optical superposition is carried out in the case of two slits for an arbitrary parameter p.

2. Theory

Consider a general case of two shifted and plane modulated wave optical signals $f_1(x)$ and $f_2(x)$:

$$g(x) = f_1(x+b)\exp(i\omega_1 x) + f_2(x-b)\exp(-i\omega_1 x), \quad (8)$$

where b is the value of shifting, $\omega_1 = 2\pi\theta/\lambda$ - is the space frequency, θ-is the angle of the incident plane wave. By virtue of linearity, the fractional Fourier image of the input signal (8) may be written as:

$$w_p(x) = u_p(x;b,\omega_1) + v_p(x;b,\omega_1), \quad (9)$$

where $u_p(x;b,\omega_1)$ and $v_p(x;b,\omega_1)$ are fractional Fourier images of shifted and modulated signals.

To calculate the FFT images, we'll use the signal distribution method [11,12]. Distribution of the optical signal (8) is the following

$$\begin{aligned} \mathrm{W}_{gg*}(x_0;\omega_0) &= \mathrm{W}_{f_1 f_1*}(x_0;\omega_0)\exp(i[\omega_0 b + x_0\omega_1]) \\ &+ \mathrm{W}_{f_2 f_2*}(x_0;\omega_0)\exp(-i[\omega_0 b + x_0\omega_1]) \\ &+ \mathrm{W}_{f_1 f_2*}(x_0 + 2b;\omega_0 - 2\omega_1) \\ &+ \mathrm{W}_{f_2 f_1*}(x_0 - 2b;\omega_0 + 2\omega_1). \end{aligned} \quad (10)$$

As it is shown in [12] the FFT distribution $W_{ww*}(x_0;\omega_0)$ may be expressed by the distribution (9) in the following manner:

$$\mathrm{W}_{ww*}(x_0;\omega_0) = \mathrm{W}_{gg*}(t_{11}x_0 + t_{12}\omega_0; t_{21}x_0 + t_{22}\omega_0), \quad (11)$$

where the linear transform of the conjugate coordinates $(x_0;\omega_0)$ of the input optical signal distribution may be described by the matrix

$$\mathbf{T}_\phi = [t_{ij}] = \begin{pmatrix} \cos\phi & -\dfrac{d_0}{k}\sin\phi \\ \dfrac{k}{d_0}\sin\phi & \cos\phi \end{pmatrix}. \quad (12)$$

The advantage of the signal distribution method is that the distribution (11) and the operator $\hat{\Im}^{-1}$ restore the FFT image [2]:

$$\hat{\Im}^{-1}[\mathrm{W}_{w_p w_p*}(0;\omega_0)] = |w_p(x)|^2. \quad (13)$$

Based on equations (10)-(13) the calculation of the FFT diffraction pattern intensity is realized by the next equation:

$$\begin{aligned} |w_p(x)|^2 &= |u_p(x;b,\omega_1) + v_p(x;b,\omega_1)|^2 = \\ &= \hat{\Im}^{-1}[\mathrm{W}_{f_1 f_1*}(t_{12}\omega_0; t_{22}\omega_0)\exp(i[t_{22}\omega_0 b + t_{12}\omega_0\omega_1])] \\ &+ \hat{\Im}^{-1}[\mathrm{W}_{f_2 f_2*}(t_{12}\omega_0; t_{22}\omega_0)\exp(-i[t_{22}\omega_0 b + t_{12}\omega_0\omega_1])] \\ &+ \hat{\Im}^{-1}[\mathrm{W}_{f_1 f_2*}(t_{12}\omega_0 + 2b;\, t_{22}\omega_0 - 2\omega_1)] \\ &+ \hat{\Im}^{-1}[\mathrm{W}_{f_1* f_2}(t_{12}\omega_0 - 2b;\, t_{22}\omega_0 + 2\omega_1)]. \end{aligned} \quad (14)$$

Two first terms of (14) form the FFT image of two shifted signals:

$$|u_p(x;b,\omega_1)|^2 = \left| u_p\left(x + b\cos\phi - \omega_1 \frac{d_0}{k}\sin\phi \right) \right|^2; \quad (15)$$

$$|v_p(x;b,\omega_1)|^2 = \left| v_p\left(x - b\cos\phi + \omega_1 \frac{d_0}{k}\sin\phi \right) \right|^2. \quad (16)$$

The obtained Eqs. (15) and (16) can be considered as generalized properties (6) and (7) of the FFT.

Accordingly, the third and fourth terms of Eq. (14) form the interference term of the FFT intensity distribution

$$\begin{aligned} i_p(x;b,\omega_1) &= u_p(x;b,\omega_1)v_p^*(x;b,\omega_1) \\ &+ u_p^*(x;b,\omega_1)v_p(x;b,\omega_1). \end{aligned} \quad (17)$$

Calculating the interference term and using Eqs. (15) and (16), we obtain the next general formula which forms the FFT image of two shifted and modulated optical signals:

$$\begin{aligned} |w_p(x)|^2 = \big| & u_p(x + B_\phi)\exp(i\Omega_\phi x) \\ & + v_p(x - B_\phi)\exp(-i\Omega_\phi x) \big|^2, \end{aligned} \quad (18)$$

The structure of this equation is equivalent to the form of an input signal (8). The generalized shifting parameter B_ϕ and modulation frequency Ω_ϕ of the FFT image can be determined by the matrix equation:

$$\begin{pmatrix} B_\phi \\ \Omega_\phi \end{pmatrix} = \mathbf{T}_\phi \begin{pmatrix} b \\ \omega_1 \end{pmatrix} \quad (19)$$

Thus, we can conclude that the forming of the FFT optical signal's (8) FFT image may be interpreted as a parallel process of the FFT image cross shifting proportionally to the value B_ϕ and the FFT image modulation by the plane wave with frequency Ω_ϕ. In this connection, the more cross shifting of the FFT images, the less their modulation and vice versa.

Using the solution (18) and matrix Eq. (19), we obtain the next condition of optical images superposition in the FFT domain:

$$B_\phi = b\cos\phi - \omega_1 \frac{d_0}{k}\sin\phi = 0, \qquad (20)$$

This Equation shows that the optical superposition is realized only in Fourier plane ($\phi = \pi/2$) for normal incidence of the plane wave ($\omega_1 = 0$). In this case, we use (2) and obtain

$$u_{p=1}(x) = C_0 F_1\left(\frac{k}{d_0}x\right); \quad v_{p=1}(x) = C_0 F_2\left(\frac{k}{d_0}x\right). \qquad (21)$$

This equation leads to formula (3).

In the general case of indirect incidence of two plane waves ($\omega_1 \neq 0$), the condition of image optical superposition takes place in the FFT domain at arbitrary value of parameter $p \neq 1$. The forming of superimposed FFT images is described by the formula

$$\begin{aligned} \left|w_p(x;0,\Omega_\phi^{\max})\right|^2 &= \left|u_p(x)\right|^2 + \left|v_p(x)\right|^2 \\ &+ u_p(x)v^*_p(x)\exp(2\Omega_\phi^{\max}x) \\ &+ u^*_p(x)v_p(x)\exp(-2\Omega_\phi^{\max}x), \end{aligned} \qquad (22)$$

where maximal amplitude of frequency modulation $\Omega_\phi^{\max}$ can be obtained from matrix Eq. (19).

Note, obtained formula (22) can be considered as a theoretical basing the principal possibility of a generalized FFT domain correlator construction in comparison with formula (3) for a classical JTC scheme.

If we introduce the FFT conjugate image of optical signal [8] $U_p(x) = \hat{\Im}^{p+1}[f(x)]$ the maximum cross shifting $B_\phi^{\max}$ is realized for zero modulation ($\Omega_\phi = 0$) of conjugate images on condition of the FFT image optical superposition, that is

$$\left|W_p(x)\right|^2 \approx \left|U_p(x + B^{\max})\right|^2 + \left|V_p(x - B^{\max})\right|^2. \qquad (23)$$

2.1.The FFT image of two slits.

Let us consider the case of two shifted and modulated plane wave unlimited slits described by the function

$$\begin{aligned} g(x) = \mathrm{rect}\left(\frac{x+b}{2a}\right)\exp(i\omega_1 x) \pm \\ \pm \mathrm{rect}\left(\frac{x-b}{2a}\right)\exp(-i\omega_1 x), \end{aligned} \qquad (24)$$

where $2a$ is the slit width, sign "$\pm$" corresponds to the case of two slits with the same and different polarities.

Let us use the next formula for a rectangular pulse [7]:

$$\begin{aligned} \mathrm{W}_{rr*}(x_0;\omega_0) = 2a\Bigg\{ &\frac{\sin(\omega_0[a + x_0/2])}{\omega_0 a}\mathrm{rect}\left(\frac{a+x_0}{2a}\right) \\ &+ \frac{\sin(\omega_0[a - x_0/2])}{\omega_0 a}\mathrm{rect}\left(\frac{a-x_0}{2a}\right)\Bigg\}. \end{aligned} \qquad (25)$$

Substituting Eq. (25) by Eq. (10), we obtain the apparent variant of the distribution of two shifted and modulated plane wave unlimited slits.

Fig.1 illustrates the results of numerical calculations of a typical distribution of two shifted and modulated plane wave unlimited slits on an informational diagram ($x_0;\omega_0$). If $\omega_1 = 0$ (Fig. 1a), the central distribution formed image is modulated by a harmonic signal. The symmetric lateral distributions determine the forming of the interference term, which has a high influence on the FFT image. If $\omega_1 \neq 0$ (Fig.1b), the central distribution of two shifted slits is changed as a result of cross shifting deformation along the x_0 axis. In this case, the orientation of the modulated band of central distribution is changed and accordingly, the lateral distribution forming interference term is mutually shifted along the line $\omega_0 = -(\omega_1/b)x_0$. The FFT distribution of such a signal can be obtained on the basis of Eqs. (11) and (12).

The FFT distributions for p=0.5 are presented on Fig.1c and 1d. As in the case of

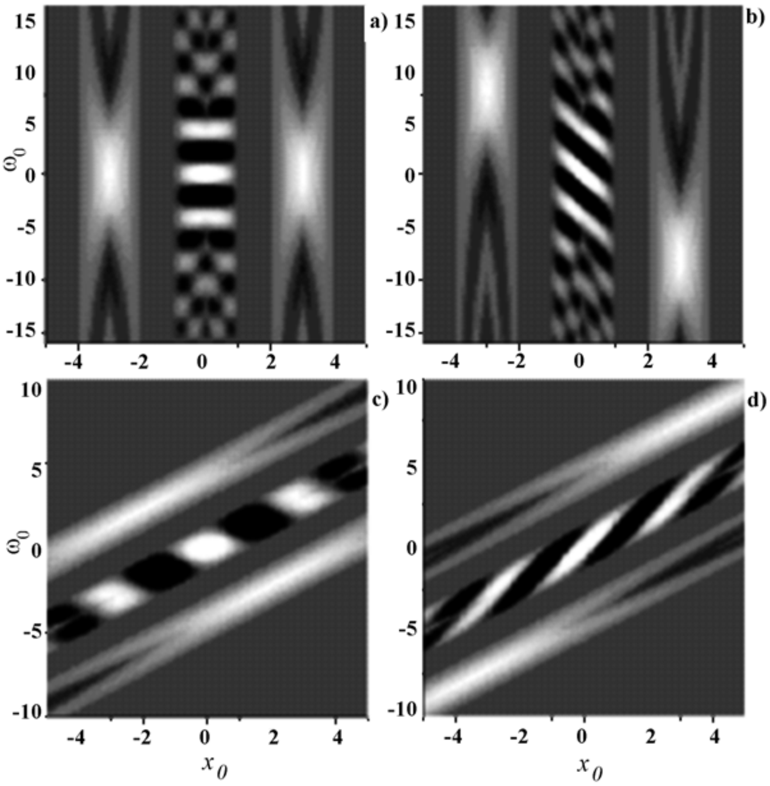

Fig. 1. Rotation on the informational diagram of the two shifted ($\beta = 2$) and modulated plane wave rectangular pulses at FFT, for Fresnel Number $F_0 = 2$: a) - $\omega_1 = 0$, $p = 0$; b) - $\omega_1 = 4$, $p = 0$; c) - $\omega_1 = 0$, $p = 1/2$; d) - $\omega_1 = 4$, $p = 1/2$.

one signal, the rotation on the informational diagram of output distribution on the angle proportional to the FFT parameter p [2] takes place in the FFT process.

Using the output distribution, we find the analytical solution of the problem in forming FFT image of two shifted slits in the next form:

$$I_p(x_a) = |u_p(x_a + B_\phi^n)|^2 + |u_p(x_a - B_\phi^n)|^2 \pm i(x_a; B_\phi^n), \quad (26)$$

where $x_a = x/a$, $B_\phi^n = B_\phi / a$ is the normalized parameter of the cross shifting.

Our calculations show that the forming of the FFT image of two shifted slits is described by the general formula:

$$|u_p(x_a \pm B_\phi^n)|^2 = \frac{2}{\pi}\int_0^{1/\sin\phi} \frac{\sin(4\pi F_0 \Omega\cos\phi[1-\Omega\sin\Omega])}{\Omega\cos\phi} \times\cos(4\pi F_0[x_a \pm B_\phi^n]\Omega)d\Omega, \quad (27)$$

where $F_0 = a^2/\lambda d_0$ is a Fresnel number. This formula is a generalization of the previous Fresnel diffration formula [7] and FFT of the self-contained slit. Note the variable path ($1 \div \infty$) in the integral formula (27) which is changed depending on the rotational angle ϕ of distribution (25) for the FFT case.

For the interference term (17), we obtain the following formula:

$$i(x_a; B_\phi^n) = \frac{2}{\pi}\int_0^{1/\sin\phi} S(\Omega; B_\phi^n)\cos\left(4\pi F_0\left[\Omega + \frac{\beta}{\sin\phi}\right]x_a\right)d\Omega + \frac{2}{\pi}\int_0^{1/\sin\phi} S(\Omega; -B_\phi^n)\cos\left(4\pi F_0\left[\Omega - \frac{\beta}{\sin\phi}\right]x_a\right)d\Omega, \quad (28)$$

where we introduce the new definition for integrand:

$$S(\Omega; \pm B_\phi^n) = 4\pi F_0(1-\Omega\sin\phi) \times\mathrm{sinc}\left(4F_0\cos\phi\left[\Omega \pm \frac{B_\phi^n}{\sin\phi\cos\phi}\right][1-\Omega\sin\phi]\right), \quad (29)$$

and where $\mathrm{sinc}(\Omega) = \sin(\pi\Omega)/\pi\Omega$ is the reading function, $\beta = b/a$.

Substitution of (27)-(29) into (26) yields the analytical solution of this problem.

3. Numerical results

Using the above – mentioned theoretical results, we investigate the regularity of the FFT images formed for two shifted and modulated plane wave unlimited slits depending on parameter p.

The typical form of the full circle parameter change p and the corresponding periodic cross displacement of the slit FFT images for the forming of the optical superimposed images is shown on Fig. 2 in three different points in the FFT domain.

Fig. 2a illustrates a typical dependence of optical image superposition in the Fourier plane at $\phi = \pi/2$ which is realized in the JTC scheme. In this figure, the optically superimposed and modulated by a harmonic signal joint transform spectrum (1) of two slits is formed in the 2F – scheme, the conjugate image of two slits is formed in the 4F – scheme.

Basing on above - mentioned results of investigation, we have theoretically proved the principal possibility to obtain the superposition of the images in an arbitrary FFT plane by the optical methods.

The typical dependencies of full circle shifting of the two unlimited slits FFT images in the case of optical superposition in points $p = 3/4$ and $p = 1/2$ are depicted in (Fig. 2b) and (Fig. 2c). Function $S(\Omega;0)$ is equivalent to under integral function (26) in accordance to the formula (28), if $B_\phi^n = 0$.

Therefore, the optically superimposed images can be expressed in an arbitrary FFT plane as:

$$I_p(x_a) = 4\cos^2\left(2\pi F_0 \frac{\beta}{\sin\phi} x_a\right)\left|u_p(x_a)\right|^2. \quad (30)$$

This equation shows that superimposed FFT images are modulated by harmonic

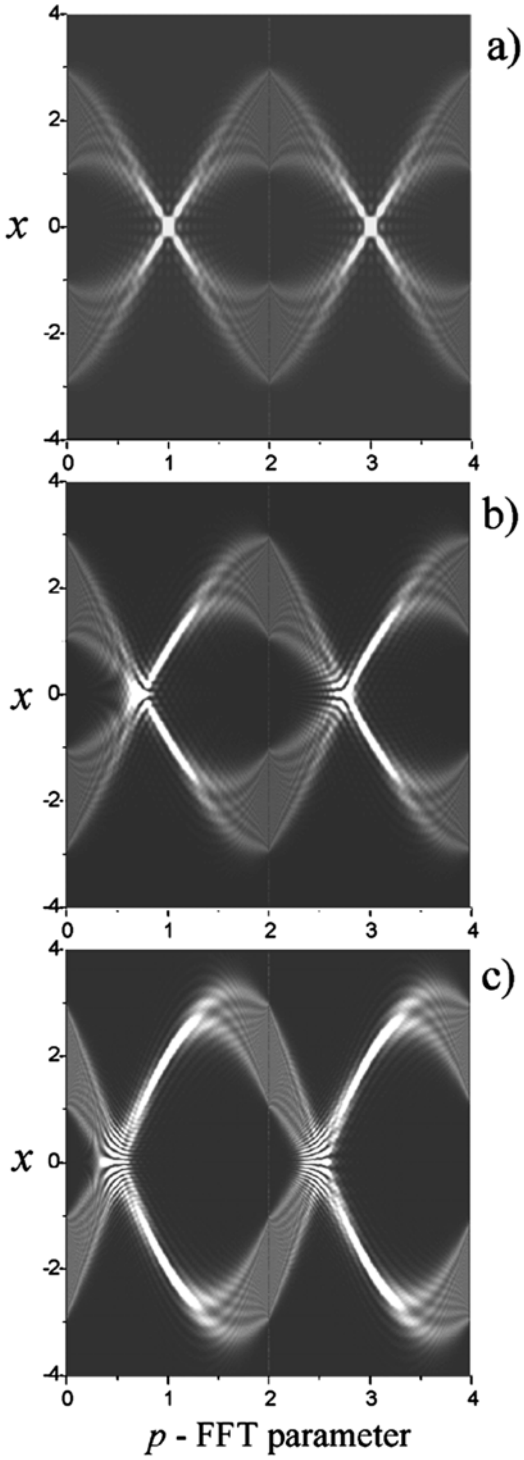

Fig. 2. Full circle of the FFT image cross shifted two infinity slits at different values of the superimposed point a) –domain of the Fourier transform; b), c) – FFT domain

interference bands as in a JTC scheme (3).

According to formula (23), the FFT conjugate images of two slits are shifted on a maximum distance $B_\phi^{\max}$. In this connection, it is

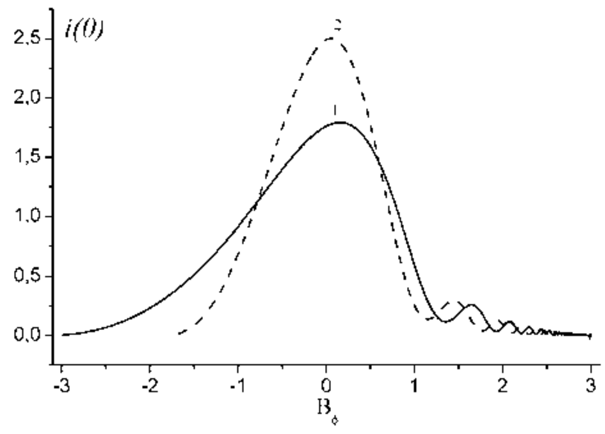

Fig. 3. Dependence "interference term *versus* B^n_ϕ": 1) - $\phi = \pi/4$; 2) - $\phi = \pi/3$; .

important to investigate their dependence, showing the interference term versus B^n_ϕ.

Fig. 3 illustrates the numerical calculations of this dependence in the different points of the FFT image superposition.

The results of numerical calculations of the FFT conjugate images of one and two rectangular slits modulated by the plane waves are shown on fig. 4 and 5. The case $p = 1/2$ is special because the FFT conjugate images (Fig. 4a and 4b) are self-similar or more exactly identical. We have shown [12], the formation of the FFT self-similar conjugate images, if $p = 1/2$, is expected for an arbitrary optical signal and it is independent on Fresnel number F_0 [2,13,14]. As it may be seen from Fig. 4c, the optical superposition of output FFT images of every slit is realized for two rectangular slits in this plane. The high contrast periodic interference bands are laid upon the superimposed image in according to formula (30).

The case of zero modulation ($\Omega_\phi = 0$) is optically realized in the plane of the FFT conjugate image forming (Fig. 4d). Such FFT images of other slits are shifted on maximum distance $2B^{\max}_{1/2}$, which is determined according to conditions (20) by an angle of inclination θ of the plane waves, basic distance $2b$ and Fresnel number F_0.

The results of numerical calculations of forming the FFT conjugate images of a rectangular slit in the plane $p = 3/4$ are shown on Fig. 5a and 5b. In this case, these images are not similar to the image $|u_{3/4}(x_a, y_a)|^2$ and this image is more like the Wigner spectrum of the slit and conjugate image $|U_{3/4}(x_a, y_a)|^2$ is more like the diffuse image of the slit. In the case of two rectangular slits, we can see the same picture of the FFT images optical superposition and maximum modulation by the periodic interference bands, as it is shown in Fig. 5c and 5b. The conjugate image of two slits is shifted to maximum distance $2B^{\max}_{3/4}$.

Thus this investigation gives a theoretical basing and the numerical confirmation of the possibility in principal of the FFT images optical superposition and the modulation of two optical signals for an arbitrary value of parameter p. The obtained results may be used for the construction of a generalized correlator in the FFT domain correlator.

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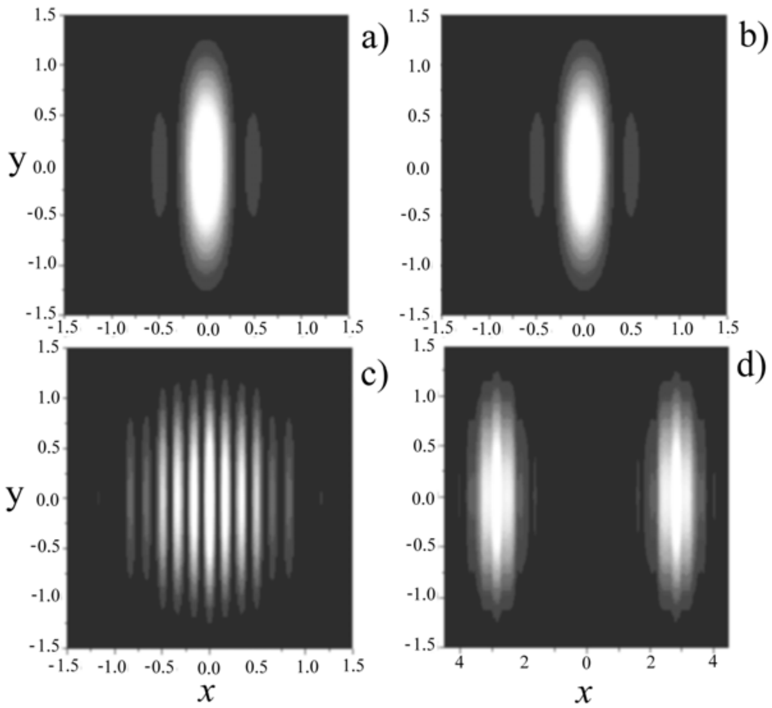

Fig.4. The rectangular slit FFT image for parameter $p=1/2$: a) - image $|u_{1/2}(x_a, y_a)|^2$ of a self-contained slit; b) – conjugate image $|U_{1/2}(x_a, y_a)|^2$ of a self-contained slit; c) - optically superimposed FFT image of two shifted ($\beta=3$) slits; d) – FFT conjugate images of two slits, shifted to the maximum.

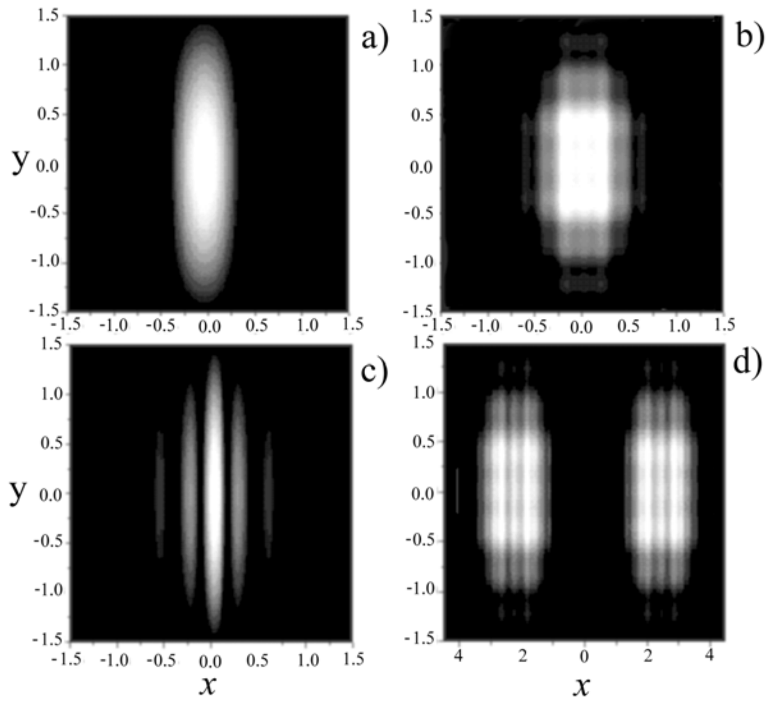

Fig.5. The rectangular slit FFT image for parameter $p=3/4$: a) - image $|u_{3/4}(x_a, y_a)|^2$ of a self-contained slit; b) – conjugate image $|U_{3/4}(x_a, y_a)|^2$ of a self-contained slit; c) - optically superimposed FFT image of two shifted ($\beta=3$) slits; d) – FFT conjugate images of two slits shifted to the maximum.

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